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CONSTRUCTION AND ANALYSIS OF A GENERALISED CONFOUNDED ASYMMETRICAL FACTORIAL DESIGN*

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Confounded asymmetrical factorial (CAF) designs achieved attention mainly in the last two decades. Many research workers worked on this aspect.

Yates (1937) introduced CAF designs by constructing designs with two level and three level factors and method of least squares was used for analysing the same. Following Yates, Kempthorne (1952) constructed CAF designs with factors at different powers of same prime number. Chakravarti (1956) made use of orthogonal arrays, Zelen (1958) introduced the concept through group divisible designs, Kishen and Srivastava (1957) used Galois field, Das (1960) made use of association with symmetric factorials and Kishen and Tyagi (1964) resorted to balanced designs for constructing CAF designs, whereas, Das and Rao (1967), Banerjee and Das (1969) and Tyagi (1971) took the aide of pairwise balanced designs for construction. Still, a general method of constructing CAF designs does not exist.

In the present paper a general method of constructing CAF designs in the line of Kishen and Srivastava (1959) is attempted.

A generalised and easy method of analysis applicable to both symmetric and asymmetric factorials is also given following the method given by Yates (1937) with Good (1958) modification.

Results and Discussion

Kishen and Srivastava (1959) constructed CAF designs $S_1 \times S_2 \times \ldots \times S_n$, ($S_1, S_2 \ldots S_n$, are prime numbers or power of prime numbers). This method requires polynomials which will take only S_i ($i = 1, \ldots, n$), values in $GF(S_i)$. In the present paper generalised method of obtaining such polynomials is given based on two lemmas.

Lemma (1) Let S be a prime number or power of a prime number and d a divisor of $s-1$. Then x^d will take only $m+1$ values in GF(s), when x assumes all the s values in GF(s), where, $s-1 = md$.

Proof Let 0, $\alpha, \alpha^2 \ldots \alpha^{s-1}$ be the s values of GF(s). Then the values x^d takes when x assumes the values of GF(s) are 0, $\alpha^d, \alpha^{2d} \ldots \alpha^{(s-1)d}$. This can be rewritten as 0, $\alpha^d, \alpha^{2d} \ldots \alpha^{md}, \alpha^{(m+1)d} \ldots \alpha^{(md)d}$. But in GF(s) $\alpha^{s-1} = 1 = \alpha^{md}$ so that the distinct values x^d can take are only 0, $\alpha^d, \alpha^{2d}, \ldots \alpha^{md}$. In other words x^d will take only $m+1$ values in GF(s).

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This lemma can be used for the construction of CAF of types 4x4x2, 7x4x3, 5x3x2 etc.

Lemma (2) If S and T are two square matrices of order $s-1$ such that, $S=p^n$, p a prime number.

$$S = \begin{bmatrix} \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{s-1} \\ \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{s-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{s-1} & \alpha_{s-1}^2 & \cdots & \alpha_{s-1}^{s-1} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} \alpha_1^{s-2} & \alpha_2^{s-2} & \cdots & \alpha_{s-1}^{s-2} \\ \alpha_1^{s-3} & \alpha_2^{s-3} & \cdots & \alpha_{s-1}^{s-3} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{s-1} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Then, T is the inverse of S.

Proof In order to prove the above result it is enough to show that ST= I.

Let $ST=R$, a square matrix of order $s-1$ and the t—th row, k—th element of R be r_{tk}. Then, two case arises.

Case I : When $t \neq k$

$$r_{tk} = (\alpha_t \alpha_k^{s-2} + \alpha_t^2 \alpha_k^{s-3} + \cdots\cdots + \alpha_t^{s-2} \alpha_k + 1) / (P-1)$$

$$\left(\left(\frac{\alpha_t}{\alpha_k}\right) \alpha_k^{S-1} + \left(\frac{\alpha_t}{\alpha_k}\right)^2 \alpha_k^{S-1} + \cdots + \left(\frac{\alpha_t}{\alpha_k}\right)^{S-2} \alpha_k^{S-1} + 1 \right) / (p-1)$$

But in GF(s),

$$\alpha_j^{S-1} = 1 \quad \text{for } j = 1, 2, \ldots\ldots\ldots, S-1 \text{ and } \frac{\alpha_t}{\alpha_k}$$

Will be an element of GF(s) say x so that

$$r_{tk} = (x + x^2 + \cdots\cdots\cdots + x^{s-2} + 1) / (p-1) = \frac{x^{s-1} - 1}{p-1} = 0$$

Since $x^{s-1} = 1$

Case II When $t = k$

$$r_{tt} = \left(\alpha_t \alpha_t^{S-2} + \alpha_t^2 \alpha_t^{S-3} + \cdots\cdots\cdots + \alpha_t^{S-2} \alpha_t + 1 \right) / (P-1)$$

$$= (S-1) / (P-1) = 1 \qquad s = p^n$$

This shows that $ST=I$.

In order to restrict the values of any polynomial to a desired number (ie., the number of levels of a factor) it is enough to solve for $a_1, a_2, \ldots, a_{s-1}$ by multiplying the matrix T with an s—1 vector with elements as the different levels of that factor.

For instance in 4x3x2 design let the polynomial be

$$a_1 x_2 + a_2 x_2^2 + a_3 x_2^3$$

T matrix is

$$T = \begin{bmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha & \alpha^2 \\ 1 & 1 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & a & \alpha^2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ a+1 \end{bmatrix}$$

The resulting polynomial is

$$\alpha x_2^2 + (\alpha+1) x_2^3$$

Analysis

The analysis of a general three factor design given here can be easily extended to design with any number of factors.

Let F_1 F_2 and F_3 be three factors at levels p, q.. and t respectively $(p \leqslant q \leqslant t)$. Arrange pqt treatment combinations in the dictionary sequence with F_1 preceeding F_2 and is succeeded by F_3. Write the sum of responses for each treatment combinations in all replications against them. Group these numbers into qt groups of p items each in the same order as they are written. These group sums will form 1/p fraction of the next column i. e., third column. In the next 1/p fraction linear contrasts corresponding to number p of these groups are written, next fraction will be formed by quadratic contrasts and next fraction cubic contrasts and so on. Orthogonal contrasts can be taken from Fisher and Yates Tables (1938). Fourth column, from the third column is obtained in a similar manner as third is obtained from the second but the grouping is done by taking pt groups of q items each. The contrasts also correspond to the number q. In a similar fashion proceeding with the number t, the fifth column is constructed from fourth. The fifth column will consists of contrasts of the final design.

Divisors of contrast squares are obtained by taking Kronecker product A of matrices M_1, M_2 and M_3 in the reverse order, where, M_1 is a p x p matrix with all elements in the first row unity and the coefficient of contrasts corresponding to P in the remaining rows. Similarly, M is a q x q matrix and M, txt matrix whose elements are taken similar to that of M_1. A will be a pqtxpqt matrix with all elements in the first row unity. Diagonal elements of AA' when multiplied with the number of replications will provide the divisors of different contrasts. While doing the entire procedure care should be taken not to violate the order.

Illustrative Example

Data on the dry weight of shoots at panicle initiation stage of rice (*Oriza sativa*) from an experiment conducted by Abdul Salam (1983) at the Tamil Nadu Agricultural University during north east monsoon season were taken. The design here was 4 x 3 x 2 factorials with treatments 4 levels of N, 3 levels of P and 2 levels of Zn.

Table 1

Dry weight of shoots at panicle initiation stage of rice, kg/ha

Treatment	Replication I	Replication II	Total
$n_0p_0z_0$	2700	2100	4800
$n_0p_0z_1$	3150	2250	5400
$n_0p_1z_0$	3250	2450	5700
$n_0p_1z_1$	3300	2750	6050
$n_0p_2z_0$	2700	3350	6050
$n_0p_2z_1$	2800	2700	5500
$n_1p_0z_0$	2850	2400	5250
$n_1p_0z_1$	2800	2700	5500
$n_1p_1z_0$	2950	3900	6650
$n_1p_1z_1$	3600	5100	8600
$n_1p_2z_0$	3300	4200	7500
$n_1p_2z_1$	3350	4300	7650
$n_2p_0z_0$	3400	3600	7000
$n_2p_0z_1$	3550	3750	7300
$n_2p_1z_0$	3700	3900	7600
$n_2p_1z_1$	4000	4200	8200
$n_2p_2z_0$	3900	3900	7800
$n_2p_2z_1$	4050	4000	8050
$n_3p_0z_0$	4200	4500	8700
$n_3p_0z_1$	4500	4800	9300
$n_3p_1z_0$	4800	5000	9800
$n_3p_1z_1$	5105	5200	10305
$n_3p_2z_0$	5200	5100	10300
$n_3p_2z_1$	5200	5100	10300

The above analysis shows that change in the levels of N and P had significant effect in shoot dry weight whereas change in the levels of Z has no effect and interactions were also not significant.

Table 2

Table of contrasts

Treatments	Contrast				
1	2	3	4	5	6
$n_0p_0z_0$	4800	25750	87350	179605	48
$n_1p_0z_0$	5250	29950	92255	80115	240
$n_2p_0z_0$	7000	31650	39550	4805	48
$n_3p_0z_0$	8700	27500	40565	11705	240
$n_0p_1z_0$	5700	33255	8350	9900	32
$n_1p_1z_0$	6850	31500	1455	900	160
$n_2p_1z_0$	7600	13450	3850	—2000	32
$n_3p_1z_0$	9800	13050	7855	9800	160
$n_0p_2z_0$	6050	13050	5900	-10010	96
$n_1p_2z_0$	7500	13500	4000	4170	480
$n_2p_2z_0$	7800	12265	—400	3290	96
$n_3p_2z_0$	10300	14800	1300	11110	480
$n_0p_0z_1$	5400	1250	—200	4905	48
$n_1p_0z_1$	5500	1050	—1800	1015	240
$n_2p_0z_1$	7300	1050	4700	—1895	48
$n_3p_0z_1$	9300	1900	5100	4005	240
$n_0p_1z_1$	6050	—545	—2500	—1900	32
$n_1p_1z_1$	8700	100	—7510	1700	160
$n_2p_1z_1$	8200	—1350	400	—1600	32
$n_3p_1z_1$	10305	1850	3770	400	160
$n_0p_2z_1$	5500	3350	200	—5010	96
$n_1p_2z_1$	7650	—1500	3090	3370	480
$n_2p_2z_1$	8050	5755	—1700	—2890	96
$n_3p_2z_1$	10300	3600	—9410	-7710	480

Anova

Source	df	SS	MS	F
Block	1	174604.69	174604.69	0.999
Treatments				
N	3	27795249.00	9265083.00	53.05**
P	2	4106563.30	2053281.60	11.76**
Z	1	501229.68	501229.68	2.87
NP	6	1136440.60	789406.76	4.52
NZ	3	145939.06	48646.35	0.28
PZ	2	374271.87	187135.93	1.07
NPZ	6	333565.61	55594.27	0.31
Error	23	4017015.84	174659 84	
Total	47	38410274.48		

Summary

The present study deals with the construction and analysis of confounded asymmetrical factorial designs. The authors have attempted to give a general method of construction with the help of two lemmas. The general method of analysis suggested by the authors can be used for symmetrical as well as asymmetrical factorial designs. An easy method of obtaining the divisions of the contrast to obtain SS in ANOVA is also explained. This method has also been illustrated through a practical example.

സംഗ്രഹം

ഏററവും അധികം തലങ്ങളുള്ള ഘടകത്തിന്റെ തലങ്ങളുടെ എണ്ണം അവിഭാജ്യ സംഖ്യയായിട്ടുള്ള കൺഫൗണ്ടഡ് അസമിത ഗുണങ്ങളുടെ നിർമ്മിതിയാണ് ഇവിടെ കൊടുത്തിരിക്കുന്നത്. അസമിത ഗുണങ്ങൾക്കും സമിത ഗുണിതങ്ങൾ ഉപയോഗിക്കാവുന്ന അപഗ്രഥന രീതിയും ഉദാഹരണ സഹിതം കൊടുത്തിരിക്കുന്നു.

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