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BALANCED N-ARY DESIGNS WITH EQUAL OR UNEQUAL BLOCK SIZES AND EQUAL OR UNEQUAL REPLICATIONS

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THESIS

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DECLARATION

I hereby declare that this thesis entitled "Balanced n-ary designs with equal or unequal block sizes and equal or unequal replications" is a bonafide record of research work done by me during the course of research and that the thesis has not previously formed the basis for the award to me of any degree, diploma, associateship, fellowship or other similar title, of any other University or Society.

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CERTIFICATE

Certified that this thesis entitled "Balanced n-ary designs with equal or unequal block sizes and equal or unequal replications" is a record of research work done independently by Smt. K.S. SUJATHA under my guidance and supervision and that it has not previously formed the basis for the award of any degree, fellowship or associateship to her.

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INTRODUCTION

INTRODUCTION

Designs are usually characterised by the nature of grouping of experimental material and the procedure of random allocation of treatments to the experimental units. A desirable property of a design is that it helps to estimate all contrasts of treatment effects and a design is said to be balanced if each of the elementary contrast can be estimated with the same variance.

The relatively simple and commonly used designs are completely randomised design, randomised block design, latin square design, split plot design etc.

In completely randomised design (Nigam and Gupta, 1979), whole of the experimental material, supposed to be homogeneous is divided into a number of experimental units depending on the number of treatments and the number of replications for each treatment. The treatments are then randomly alloted to the units in the entire material.

The variance of the difference between i^{th} and j^{th} treatment mean is given by $\sigma^2(\frac{1}{r_i} + \frac{1}{r_j})$ where r_i and r_j are the number of replications for the i^{th} and j^{th} treatments. If $r_i = r_j = r$ variance of the elementary contrast is given by $\frac{2\sigma^2}{r}$ and hence completely randomised design is balanced if each of the treatments has been replicated the same number of times.

In order to control variability in one direction in the experimental material it is desirable to divide the experimental units into homogeneous group of units known as blocks. The treatments are randomly allocated separately to each of these blocks. This procedure gives rise to a design known as Randomised Block Design, which can be defined as an arrangement of v treatments in r blocks such that each treatment occurs precisely once in each block (Nigam and Gupta, 1979).

The estimated variance of the difference between i^{th} and j^{th} treatment mean is given by $\frac{2\sigma^2}{r}$, i.e., every elementary contrast can be estimated with the same variance and hence Randomised Block Design is always balanced.

To control two way heterogenity in the experimental material we use a design known as Latin square designs. In such designs, two restrictions are imposed by forming blocks in two directions, rowwise and columnwise. Treatments are allocated in such a way that every treatment occurs once in each row and each column (Cochran and Cox, 1957). Variance of an elementary contrast is $\frac{2\sigma^2}{r}$ and hence latin square design is always balanced.

By using latin square designs treatment effects can be estimated by eliminating more sources of variation. A graeco Latin Square is a Latin Square Design in which a third orthogonal effect has been accounted for by a classification represented by greek letters. As in the case of Latin square design, Graeco Latin Square is also balanced.

The concept of Graeco Latin Square can be extended to obtain hyper-graeco latin square. These are obtained by superimposing three or more orthogonal latin squares. These designs too are balanced.

When different factors influence a character under study, it is desirable to test different combinations of the factors at various levels. Such experiments are called factorial experiments. In an unconfounded factorial experiment every main effect/interaction can be estimated with the same variance and hence factorial experiment is balanced if there is no confounding of effects.

In a factorial experiment when the number of factors and/or levels of factors increases, the number of treatment combinations increases very rapidly and it is not possible to accommodate all these treatment combinations in a single homogeneous block. A method of overcoming this difficulty is to adopt the principle of confounding. If the same set of interactions get confounded in all replications, it is called total confounding whereas if different sets of interactions get confounded in different replications it is called partial confounding. In the case of partial confounding, particular set of interactions confounded in a replication can be estimated from the information obtained on these effects from the remaining replications in which these effects are not confounded. A factorial experiment is called a balanced factorial experiment if the relative loss of information in each of the single degree of freedom belonging to any partially confounded effect is same (Nigam and Gupta, 1979).

In field experiments, sometimes some factors require large experimental units while some others require only plots of smaller size. In such situations split plot designs are used. A split plot design using an Randomised Block Design for the first set of treatments (called main plot treatments) is obtained by alloting the main plot treatments at random to the whole plots of a block and then randomising the second set of treatments (called sub plot treatments) within each whole plot. This enables to compare the effects of the sub plot treatments and test presence of interactions of the whole plot treatments with sub plot treatments more efficiently than testing the difference in the main effect of the main plot treatments. This is because of the fact that main effect of the whole plot treatments get confounded with block effects (Das and Giri, 1979). In this design every main plot treatment is estimated with the same variance. Further every sub plot treatment difference is estimated with equal variance. That is to say there is balance over the estimates of the different categories of elementary contrasts.

Strip plot design is a design analogous to the split plot arrangement, in which two different sets of treatments can be tried in large plots with one set of plots superimposed over the other set at right angles. A block may be divided into strips in one direction to be allotted to one set of treatments and into another set of stripe, in a direction at right angles to the first, to be allotted to the second set of treatments. The plots formed by the interaction of the strips may be further split or the entire primary strips belonging to one set may themselves be divided into further narrower strips for accommodating a yet further set of treatments (Panse and Sukhatme, 1954). We observe that elementary contrasts of treatments in each strip are estimated with equal variance though there may be difference in the variances of the estimates of elementary contrasts of the two categories of treatments.

When number of treatments are large all treatments cannot be accommodated in a block. Designs in which blocks do not accommodate all treatments are to be preferred in such a context. A design in which there is at least one block which does not accommodate all the treatments is called an incomplete block design. An incomplete block design which gives equal information on all elementary contrasts can be called a balanced incomplete block design.

A balanced incomplete block design (Yates, 1936) is an arrangement of v treatments in b blocks each of size $K < v$ such that every treatment is replicated r times in r distinct blocks and every pair of treatments occur together in λ blocks.

BIB designs have the property that every elementary contrast is estimated with same precision. These designs require at least as large a number of blocks as the number of treatments. Further the number of replications cannot be less than the size of the block. Bose and Nair (1939) introduced partially balanced incomplete block (PBIB) designs where the property that every elementary contrast is estimated with equal precision is relaxed (Nigam and Gupta, 1979).

In animal experiments as also in perennial crop experiments largest variation is due to subjects themselves. One way to get over this variation is to test the different treatment on every animal. To do this we switch the animal to different treatments from time to time. The experiment is thus called switch over, change over or cross over trial. As the character measured changes from period to period and from animal to animal, in switch over trials, every treatment should be tested on every animal and in every period equally frequently.

A problem in switch over experiment is that except for the first period the yield is not the direct effect of the treatment applied in that period but also due to the residual or carry over effect of a treatment applied during the previous period. If the design used is such that every treatment is preceded by every other treatment exactly the same number of times it is said to be balanced with respect to the residual effects.

The designs discussed above share some common properties. Except for the completely randomised design they are all equi-replicate, have equal block sizes and are binary (A design in which a treatment occurs in a block at most once is said to be binary. It is said to be proper if all its blocks are of the same size). Further similar elementary comparisons of treatments have equal variances. That is to say such differences are balanced. It appears that this idea of balance is woven into the fabric of design of experiments. However, the experimental situations may not be favourable for the choice of equal block size; nor can it always permit equal replication. It may not also be possible to replicate each treatment equally frequently in each block. Above all these the binary nature of the design may have to be sacrificed. Examples of such situations are many. In experiments on animals the litter sizes (blocks) are invariably different and we may make use of all animals in a litter; or as is pointed out by Pearce (1964) while comparing a new source of variety with an old one equal replication of treatments may serve as an impedement. And in spite of these handicaps the experimenter would like to have equal information on all elementary treatment comparisons as if he were to use an ordinary design like randomised block. These situations, as pointed out above are not rare in practice. Designs suitable for experiments under such situations are therefore to be made available. This calls for the construction of n-ary balanced designs with equal or unequal replications in blocks of equal or unequal sizes.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

Balanced n-ary designs were introduced by Tocher (1952) as a generalisation of balanced incomplete block design. His designs are proper with constant block size and generally equireplicate. Tocher defined (proper) n-ary design as an arrangement of v treatments in b blocks each of size k such that each treatment occurs r times in the whole design and $\sum n_{ij} n_{im} (j \neq m)$ is a constant, where n_{ij} denotes the number of times the i^{th} treatment occurs in the j^{th} block and can take values from 0 to $(n-1)$. Some balanced ternary designs (having frequencies 0,1,2) in which the treatments were not replicated the same number of times were also suggested by him. As an illustration one such design is reproduced below.

Ternary designs with $v=6$, $b=6$, $k=4$, $\lambda=2$.

2	1	1	0	0	0
0	2	0	1	0	1
0	0	2	1	0	1
1	0	0	2	1	0
0	1	1	0	2	0
1	0	0	0	1	2

In order to understand the full significance of the balanced designs in general a few preliminary results are required. They are derived below.

Consider v treatments arranged in b blocks. Assume that n_{ij} is the number of times the i^{th} treatment occurs in the j^{th} block (i=1, 2, ...,v). Let the size of the j^{th} block be k_j (j= 1, 2, ...,b). Then $N=(n_{ij})$ is called the incidence matrix of the design. For the sake of generality we assume that the i^{th} treatment is replicated r_i times.

The linear model usually taken for the analysis of this design is

$$Y_{ijk} = \mu + t_i + \alpha_j + E_{ijk}$$

$i = 1, \ldots, v$; $j = 1, \ldots, b$; $k=1, \ldots, n_{ij}$.

where Y_{ijk} is the yield of the i^{th} treatment from the k^{th} plot of the j^{th} block, μ is general mean, t_i is effect of i^{th} treatment, α_j is effect of j^{th} block and E_{ijk} are independent normal variables with expectation zero and variance σ^2.

The normal equations for estimating the treatment effects are obtained by the method of least squares. This theory has been developed by Gauss Markoff which states that, "the unbiased linear estimate of minimum variance of any parameter is that given by the method of least squares".

The normal equations are

$$(2.1) \quad Y_{\cdot\cdot\cdot} = n\hat{\mu} + \sum_i n_{i\cdot}\hat{t}_i + \sum_j n_{\cdot j}\hat{\alpha}_j$$

$$(2.2) \quad Y_{i\cdot\cdot} = n_{i\cdot}\hat{\mu} + n_{i\cdot}\hat{t}_i + \sum_j n_{ij}\hat{\alpha}_j$$

$$(2.3) \quad Y_{\cdot j\cdot} = n_{\cdot j}\hat{\mu} + \sum_i n_{ij}\hat{t}_i + n_{\cdot j}\hat{\alpha}_j$$

From (2.3)

$$\hat{\alpha}_j = \frac{Y_{\cdot j\cdot}}{n_{\cdot j}} - \hat{\mu} - \frac{1}{n_{\cdot j}}\sum_i n_{ij}\hat{t}_i$$

substituting in (2.2) and rearranging

$$Y_{i\cdot\cdot} - \sum_j \frac{n_{ij}Y_{\cdot j\cdot}}{n_{\cdot j}} = n_{i\cdot}\hat{t}_i - \sum_j \frac{n_{ij}}{n_{\cdot j}}\sum_i n_{ij}\hat{t}_i$$

If the L.H.S. is defined as Q_i,

$$(2.4) \quad Q_i = \left(n_{i\cdot} - \sum_j \frac{n_{ij}^2}{n_{\cdot j}}\right)\hat{t}_i - \sum_{\substack{i=1 \\ i\neq i'}}^{v} \frac{n_{ij}n_{i'j}}{n_{\cdot j}}\hat{t}_{i'}$$

Evidently $\sum_i Q_i = 0$ and (2.4) can be written as,

$$(2.5) \quad Q_i = \lambda_{ii}\hat{t}_i + \sum_{\substack{i \\ i\neq i'}} \lambda_{ii'}\hat{t}_{i'}$$

It is easy to show that the sum of elements of matrix on any row of the R.H.S. of (2.5) is zero. The same will be true of the columns of the matrix. Since the sum of L.H.S. and R.H.S. are identically zero, the equations are not independent. Hence to obtain an unique solution we may impose a condition $\sum_i t_i = 0$.

In matrix notation equation (2.5) can be written as,

(2.6) $Q = C\hat{t}$, (Kempthorne, 1952), where

$$C_{ii} = n_{i.} - \sum_j \frac{n_{ij}^2}{n_{.j}}$$

$$C_{ii'} = - \sum_j \frac{n_{ij} n_{i'j}}{n_{.j}}$$

$$t' = (t_1, \ldots, t_v)$$

$$Q' = (Q_1, \ldots, Q_v)$$

The matrix on the R.H.S. of (2.6) is called C-matrix or coefficient matrix of the design. If we denote $n_{i.}$, the number of replications of the i^{th} treatment, by r_i; $n_{.j}$ the size of the j^{th} block, by k_j, and define R_i

$R_i = \text{diag}(r_1, \ldots, r_v)$, $k = \text{diag}(k_1, \ldots, k_b)$ then,

(2.7) $C = R - NK^{-1}N'$

Since

$$V(Q_i) = (n_{i.} - \sum \frac{n_{ij}^2}{n_{.j}})\sigma^{-2}$$

(2.8)

$$\text{Cov}(Q_i, Q_{i'}) = - \sum \frac{n_{ij} n_{i'j}}{n_{.j}} \sigma^{-2},$$

the dispersion matrix of Q is $\sigma^{-2}C$ where σ^{-2} is the intra block error.

(2.9) That is, $V(Q) = C\sigma^{-2}$

Obviously C is symmetric and sum of elements of any row or column is zero. Hence the rank of C is never greater than (v-1). If the rank of C is (v-1) the design is said to be connected.

Connectedness was originally defined by R.C. Bose.

A treatment and a block are said to be associated if the treatment occurs in the block. Two treatments are said to be connected if it is possible to pass from one to the other through a chain of associations between treatments and blocks. A design is said to be connected if every pair of treatments in it are connected. It is not difficult to show that a necessary and sufficient condition for a design to be connected in Boses' sense is that the rank of C matrix is (v-1).

We shall now show that only contrasts of treatment effects are estimable. By a contrast of treatment effects we mean a linear function of these effects such that the sum of coefficients is zero. From the theory of linear estimation of L't is estimable, there exists a ρ such that $C\rho = L$ thereby showing that rank C = rank (C,L).

i.e. L depends on columns of C

i.e. L is a linear function of columns of C.

Therefore, the sum of elements of L is zero, thus showing that L't is a contrast. Thus every estimable linear function of t is a contrast.

It is easy to deduce the following results.

Result 2.1. The number of independent estimable treatment contrasts is $(v-t)$, where $v-t$ is the rank of C.

The number of linearly independent solutions of L can be at most $(v-t)$, the rank of C.

If $t=1$, $(v-1)$ linearly independent contrasts are estimable.

Result 2.2. If the rank of C is $(v-1)$ all the $(v-1)$ linearly independent elementary contrasts are estimable.

Given v treatments there are $\frac{1}{2}v(v-1)$ distinct elementary contrasts of the type (t_i-t_j). Among these, only $(v-1)$ are linearly independent.

We shall now derive the expression for variance of an estimable linear function.

Let $L'\hat{t}$ be the estimate of an estimable linear function of $L't$. From $C\hat{t} = Q$ it follows that there exists a ρ such that $C\rho = L$.

$$
\begin{aligned}
\therefore \quad V(L'\hat{t}) &= E(L'\hat{t}\hat{t}'L) \\
&= E(\rho' C'\hat{t}\hat{t}'C\rho) \\
&= E(\rho' (QQ') \rho), \quad (\because \; C\hat{t} = Q) \\
&= \rho' C\rho\sigma^2 \\
&= \rho' L\sigma^2 \\
&= L'\rho\sigma^2
\end{aligned}
$$

A useful inequality derived by Sylvain Ehrenfeld (1955) is as follows.

Theorem 2.1. If $L't$ is an estimable contrast, then

$\frac{L'L\sigma^2}{\lambda \max.} \leq V(L'\hat{t}) \leq \frac{L'L\sigma^2}{\lambda \min.}$ where λ max and λ min

are the maximum and minimum values of the characteristic roots of C.

Proof. We shall assume that rank of C is p. Since C is a real symmetric matrix there exists an orthogonal matrix S such that $S'CS = \text{diag}(\lambda_1, \ldots, \lambda_p, 0 \ldots 0)$

$$\therefore \rho' C\rho = \rho' SS'CSS'\rho \text{ ; for } SS' = S'S = I$$

$$= \rho' S \,\text{diag}(\lambda_1, \ldots, \lambda_p, 0 \ldots 0) S'\rho$$

If we denote $\rho' S = (u_1, u_2, \ldots, u_v)$

$$= \sum u_i^2 \lambda_i$$

$$\rho' C\rho = \sum_i Y_i^2 \quad (Y_i = u_i\sqrt{\lambda_i})$$

$$\text{Now } L'L = (C\rho)'(C\rho)$$

$$= \rho' CC\rho$$

$$= \rho' SS' C SS' C SS' \rho$$

$$= \rho' S(S'CS)\,(S'CS)\,S'\rho$$

$$= \rho' S\,(\mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_p, 0 \ldots 0)$$

$$(\mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_p, 0 \ldots 0)\,S'\rho$$

$$= (u_1, u_2, \ldots, u_v)\,\mathrm{diag}\,(\lambda_1^2, \ldots, \lambda_p^2, 0 \ldots 0) \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_v \end{pmatrix}$$

$$= \sum \lambda_i^2 u_i^2$$

$$= \sum_i \lambda_i Y_i^2$$

$$\therefore \quad \frac{\rho' C \rho}{L'L} = \frac{\sum Y_i^2}{\sum \lambda_i Y_i^2}$$

The R.H.S. is the inverse of weighted mean of the non-zero roots of C. Hence it should lie between

$$\frac{1}{\lambda_{\max.}} \quad \text{and} \quad \frac{1}{\lambda_{\min.}}$$

$$\therefore \quad \frac{1}{\lambda_{\max.}} \leq \frac{\rho' C \rho}{L'L} \leq \frac{1}{\lambda_{\min}}\,.$$

Thus remembering that $V(L'\hat{t}) = \rho' C \rho \sigma^{-2}$ we get.

$$(2.10) \qquad \frac{L'L\sigma^2}{\lambda_{\max}} \leq V(L'\hat{t}) \leq \frac{L'L\sigma^2}{\lambda_{\min}}$$

Corollary 2.1. If $L'\hat{t} = (\hat{t}_i - \hat{t}_j)$ is estimable

$$\frac{2\sigma^2}{\lambda_{\max}} \leq V(\hat{t}_i - \hat{t}_j) \leq \frac{2\sigma^2}{\lambda_{\min}}$$

Corollary 2.2. Every estimable elementary contrast will have the same variance if all non-zero characteristic roots of C matrix are equal and the common variance is given by $\frac{2\sigma^2}{\lambda}$ where λ is the common value of the non-zero characteristic roots.

We shall now focus our attention on the balance of a design.

A design is said to be balanced if the best linear unbiased estimate of every elementary contrast has the same variance.

We now prove

Theorem 2.2. If λ_i, $i = 1, 2, \ldots, (v-1)$ are the roots of the C matrix of a connected design, the average variance of an elementary contrast is $\frac{2\sigma^2}{v-1} \sum_{i=1}^{v-1} \frac{1}{\lambda_i}$

Proof. Since C is symmetric matrix of order v of the design.

$$\left| C - \lambda I(v) \right| = \lambda(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_{v-1})$$

Since the sum of elements of each row of C is zero the normalised characteristic vector corresponding to the zero root of C is $\frac{1}{\sqrt{v}}$ $E(v,1)$ where $E(p,q)$ stands for a p x q matrix whose elements are all unity.

Then for any $a \neq 0$,

$$\frac{E(1,v)}{\sqrt{v}}\left[C+aE(v,v)\right]\frac{E(v,1)}{\sqrt{v}} = av$$

Hence $C+aE(v,v)$ is nonsingular. Its roots are av, $\lambda_1, \lambda_2, \ldots, \lambda_{v-1}$ which have characteristic vectors same as those of C. Let the characteristic vectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_{v-1}$ be $L_1, L_2, \ldots, L_{v-1}$

Then $\left[C+aE(v,v)\right]^{-1} = \frac{1}{av^2}E(v,v)+\sum_i \frac{1}{\lambda_i}L_iL_i'$

So that $E(v,v)\left[C+aE(v,v)\right]^{-1} = \frac{E(v,v)}{av}$

Also $\hat{t} = \left[C+aE(v,v)\right]^{-1}Q$

For, $C\hat{t} = C\left[C+aE(v,v)\right]^{-1}Q$

$= \left[C+aE(v,v)-aE(v,v)\right]\left[(C+aE(v,v)\right]^{-1}Q$

$= \left[I(v)-\frac{1}{v}E(v,v)\right]Q$

$= Q$, because $E(v,v)\,Q = 0$

$\therefore\ \hat{t} = \left[C+aE(v,v)\right]^{-1}Q$ can be taken as a solution of t.

$V(\hat{t}) = \left[C+aE(v,v)\right]^{-1}V(Q)\left[C+aE(v,v)\right]^{-1'}$

$= \left[C+aE(v,v)\right]^{-1}C\left[C+aE(v,v)\right]^{-1'}\sigma^2$

$= \left[\left[C+aE(v,v)\right]^{-1}-\frac{1}{av^2}E(v,v)\right]\sigma^2$

after simplification.

$\therefore$ The variance of $L'\hat{t}$, a best unbiased estimate of a contrast is

$$V(L'\hat{t}) = L'\left[\left[C+aE(v,v)\right]^{-1} - \frac{1}{av^2} E(v,v)\right] L\sigma^2$$

$$= L'\left[C+aE(v,v)\right]^{-1} L\sigma^2$$

Denoting the ij^{th} element of $\left[C+aE(v,v)\right]^{-1}$ by u_{ij}, the above relation gives

$$V(\hat{t}_i-\hat{t}_j) = (u_{ii}+u_{jj}-2u_{ij})\sigma^2$$

$\therefore$ Average variance of an elementary contrast is

$$\frac{1}{v(v-1)} \sum_{i} \sum_{i \neq j} (u_{ii}+u_{jj}-2u_{ij})\sigma^2$$

$$= \frac{2\sigma^2}{v(v-1)} \left(v \sum_{i=1}^{v} u_{ii} - \sum_{i,j=1}^{v} u_{ij}\right)$$

$$= \frac{2\sigma^2}{v(v-1)} \left[v \text{ trace } U - \frac{1}{v} \text{ trace } E(v,v)\, UE(v,v)\right]$$

where $U = \left[C+aE(v,v)\right]^{-1}$

$$(2.11) \qquad = \frac{2\sigma^2}{v-1} \sum_{i=1}^{v-1} \frac{1}{\lambda_i}$$

after simplification. Rao (1958) and Kempthorne (1956) have proved these results though by different methods.

Thus the average variance of an elementary contrast in a connected design is proportional to the inverse of the Harmonic mean of the non-zero characteristic roots of the C-matrix.

Kempthorne (1956) has shown that the efficiency factor of a design is r times the harmonic mean of the latent roots of the reduced intra block normal equations including the root which is always zero. Let us now define the most efficient design as one which minimises the average variance of an elementary contrast. We confine ourselves to such designs for which the sum of roots of C is a constant. To obtain the most efficient design we have therefore to minimise $\sum \frac{1}{\lambda_i}$ subject to the condition that $\sum \lambda_i$ = a constant (say D). Thus we have to minimise unconditionally

$$\sum_{i=1}^{v-1} \frac{1}{\lambda_i} + \mu(\lambda_i - D) \text{ w.r.t. } \mu \text{ and } \lambda_i$$

where μ is an Lagrangian multiplier. This leads to the result that λ_i is a constant. Therefore those designs which have got all the (v-1) non-zero roots of C constant are most efficient provided $\sum \lambda_i$ is a constant. Evidently balanced incomplete block design is most efficient among the binary designs. This result is due to Kshirsagar (1958).

Another result which will be of immense help in the discussion of balanced design is theorem 2.3 due to Roy and Laha (1957).

Theorem 2.3. A necessary and sufficient condition for a symmetric matrix of order v to have $(v-1)$ roots equal is that its diagonal elements are equal and the characteristic vector corresponding to the root of multiplicity 1 is $\frac{E(v,1)}{\sqrt{v}}$

Proof. Let the matrix A be of the form

$$(2.12) \quad A = (a-b)I(v)+bE(v,v)$$

Then

$$|A-\lambda I| = \begin{vmatrix} a-\lambda & b & \cdot & \cdot & \cdot & b \\ b & a-\lambda & \cdot & \cdot & \cdot & b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & \cdot & \cdot & \cdot & a-\lambda \end{vmatrix} = 0$$

gives roots as $a+(v-1)b$, and $(a-b)$ with multiplicities 1 and $(v-1)$ respectively. Evidently the characteristic vector corresponding to $a+(v-1)b$ is $\frac{1}{\sqrt{v}}E(v,1)$

conversely let there be a symmetric matrix of order v such that it has $(v-1)$ roots each equal to λ_2 and one root equal to λ_1, the latter having characteristic vector $\frac{1}{\sqrt{v}}E(v,1)$

let the characteristic vectors of the matrix be

$$Z_1 = \frac{1}{\sqrt{v}} E(v,1),\ Z_2,\ \ldots,\ Z_v$$

$(Z_1, Z_2, \ldots, Z_v) = N$ is an orthogonal matrix.

So that $\sum_1^v Z_i Z_i' = I(v)$,

from where

$$(2.13) \qquad \sum_2^v Z_i Z_i' = I(v) - \frac{1}{v} E(v,v).$$

The symmetric matrix itself is equal to

$$\lambda_1 Z_1 Z_1' + \lambda_2 \sum_2^v Z_i Z_i'$$

$$= \lambda_2 I(v) + \frac{\lambda_1 - \lambda_2}{v} E(v,v)$$

and this has diagonal elements $\lambda_2 + \frac{\lambda_1 - \lambda_2}{v}$

and off diagonal elements $\frac{\lambda_1 - \lambda_2}{v}$

The above result will be used to prove the theorem given below.

It is easy to infer that for a proper equireplicate binary design to be balanced is that NN' has all diagonal elements equal and all offdiagonal elements equal.

Theorem 2.4. A necessary and sufficient condition for every elementary contrast to have the same variance is that the design is connected and that all the non-zero roots of C are equal.

Proof. Suppose the design is balanced. That is every elementary contrast has the same variance which necessarily should be $\frac{2\sigma^2}{v-1}\sum_{1}^{v-1}\frac{1}{\lambda_i}$

Now, $V(\hat{t}_i-\hat{t}_j) = V\ (\hat{t}_i-\hat{t}_k)-(\hat{t}_j-\hat{t}_k)$

which leads to the result covariance between two elementary contrasts having a common element is

$$(2.13)\qquad \frac{\sigma^2}{v-1}\sum_{i=1}^{v-1}\frac{1}{\lambda_i}$$

Let γ_i be the characteristic vector corresponding to a non-zero root λ_i of C. Then $E(1,v)\ \gamma_i=0$ and $\gamma_i{}'\hat{t}$ is a contrast.

$$(2.14)\qquad V(\gamma_i{}'\hat{t}) = \gamma_i{}'\left[(C+aE(v,v)\right]^{-1}\gamma_i\sigma^2$$

$$= \frac{\sigma^2}{\lambda_i}$$

$\gamma_i{}'$ belongs to the vector space of the vectors of contrasts. The (v-1) elementary independent contrasts belong to this vector space.

$\therefore$ $\gamma_i'\hat{t}$ is expressible in terms of $(v-1)$ independent elementary contrasts.

Take,

$$\gamma_i'\hat{t} = a_1(\hat{t}_1-\hat{t}_v) + a_2(\hat{t}_2-\hat{t}_v) + \ldots + a_{v-1}(\hat{t}_{v-1}-\hat{t}_v)$$

$$= a_1\hat{t}_1 + a_2\hat{t}_2 + \ldots + a_{v-1}\hat{t}_{v-1} - (\sum_1^{v-1} a_i)\,\hat{t}_v$$

As $\gamma_i'\gamma_i = 1$, $\sum_1^{v-1} a_i^2 + (\sum_1^{v} a_i)^2 = 1$

(2.15) Now

$$V(\gamma_i'\hat{t}) = \sum_1^{v-1} a_i^2 V(\hat{t}_i-\hat{t}_v) + \sum_{\substack{i,j=1\\ i\neq j}}^{v-1} \mathrm{Cov}\,(t_i-t_v,\ t_j-t_v)a_i a_j$$

$$= \frac{\sigma^2}{v-1}\sum_1^{v-1}\frac{1}{\lambda_i}$$

Combining (2.14) and (2.15),

(2.16) $$\frac{\sigma^2}{\lambda_i} = \frac{\sigma^2}{v-1}\sum_{j=1}^{v-1}\frac{1}{\lambda_j},$$ (Kshirsagar, 1958)

for all i for which $\lambda_i \neq 0$.

Thus every $\lambda_i \neq 0$ is the same

or C has all the $(v-1)$ non-zero characteristic roots equal.

Conversely if $\lambda_1, \lambda_2, \ldots, \lambda_{v-1} = \lambda$ (say) then we have

from the relation (2.10) that $\frac{2\sigma^2}{\lambda} \leq v(\hat{t}_i-\hat{t}_j) \leq \frac{2\sigma^2}{\lambda}$

$\therefore$ $V(t_i - t_j)$ is the same and is equal to $\frac{2\sigma^2}{\lambda}$ and hence the design is balanced and in this case by theorem (2.3) C assumes the form (2.12).

The above theorem was proved by Rao (1958) in a different manner. Rao also proved as a corollary that if a binary balanced design is proper, then it must be equireplicate.

Atiquallah (1961) has derived a necessary and sufficient condition for a connected design to be balanced as a natural extension of a result by Tocher (1952) and of Thompson (1956). It appears to be simpler than the generalisation given by Rao (1958). He has also derived an expression for calculating the efficiency factor of a connected design. The Fishers inequality that $b \geq v$ for a balanced incomplete block design with v treatments and b blocks is also shown to be true for a wider class of binary designs, similar to the balanced incomplete block designs with blocks of different sizes.

The following theorem was proved.

Theorem 2.5. A necessary and sufficient condition for a connected design to be balanced is that every $\hat{t}_j (j=1,\ldots,v)$ is estimated with the same variance and every pair $\hat{t}_j$, $\hat{t}_{j'}$ with the same covariance.

Before proving the theorem we have to establish the result $\sum_j \text{var}(t_j) = \frac{\sigma^2}{H}(v-1)$ which is used for the necessary part.

Proof. We have $C\hat{t} = Q$. Let (θN) be the orthogonal matrix whose columns are the characteristic vectors of C with θ = a column vector with each element $\frac{1}{\sqrt{v}}$

$$\begin{pmatrix} \theta^1 \\ N^1 \end{pmatrix} Ct = \begin{pmatrix} \theta^1 \\ N^1 \end{pmatrix} Q$$

$$(2.7) \quad \text{i.e.} \quad \begin{pmatrix} \theta^1 Ct \\ N^1 Ct \end{pmatrix} = \begin{pmatrix} \theta^1 Q \\ N^1 Q \end{pmatrix}$$

But $N^1Ct = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{v-1}) N^1 t = N^1 Q$

Denoting $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{v-1})$ by D, we get

$$N^1 t = D^{-1} N^1 Q$$

or $\quad NN^1 t = ND^{-1}N^1Q$

i.e. $(I_v - \frac{1}{v} E(v,v)) = ND^{-1}N^1Q$

i.e. $\quad \hat{t} = ND^{-1}N^1Q$

and

$$(2.18) \quad V(\hat{t}) = ND^{-1}N^1CND^{-1}N^1\sigma^2$$

$$= ND^{-1}N^1\sigma^2$$

In a connected design average variance of an elementary contrast is $\frac{1}{v(v-1)} \sum_{j \neq j'} (t_j - t_{j'}) = \frac{2\sigma^2}{H}$ where H is the harmonic mean of the non-zero roots of C.

From (2.18)

$$\sum_j V(t_j) = \sigma^2 \operatorname{tr} ND^{-1}N'$$

$$= \sigma^2 \sum \frac{1}{r_i} \quad \text{for } N'N = I_{v-1}$$

(2.19) $$= \frac{(v-1)\sigma^2}{H}$$

Let the design be balanced. Then the C-matrix has (v-1) non-zero equal roots each equal to $\frac{1}{H}$

$$V(t_1-t_2)+V(t_1-t_3) + \ldots + V(t_1-t_v) = (v-1)v(t_1)$$

$$+ \sum_{j=2}^{v} v(t_j) - 2 \sum_{j=2}^{v} \mathrm{Cov}(t_1,t_j)$$

$$= (v-1)V(t_1) + \sum_{j=2}^{v} V(t_j) - 2\,\mathrm{Cov}(t_1,-t_1)$$

(2.20) i.e. $$V(t_1-t_2) + \ldots + V(t_1-t_v) = V(t_1) + \sum_{j=1}^{v} V(t_j)$$

In the same way

$$V(t_2-t_1)+V(t_2-t_3)+\ldots+V(t_2-t_v) = V(t_2) + \sum_{j=1}^{v} V(t_j)$$

Since the design is balanced every elementary treatment contrast has the same variance and therefore from (2.20) and (2.21) it follows that $V(t_1)=V(t_2)= \ldots =V(t_v)$.

Thus from (2.18) $V(t_j) = \frac{\sigma^2}{H} \frac{(v-1)}{v}$

$$= \frac{\sigma^2}{H}(1- \frac{1}{v}) \text{ and}$$

$V(t_j - t_j1) = 2v(t_j) - 2 \text{ Cov}(t_j, t_j1) = \frac{2\sigma^2}{H}$

Hence Cov $(t_j, t_{j'}) = \frac{\sigma^2}{H} - \frac{\sigma^2}{H}(1-\frac{1}{v})$

$$= \frac{\sigma^2}{H} \times \frac{1}{v}$$

.˙. Variance co-variance matrix of t is, except for σ^2

$$\frac{1}{H}\begin{bmatrix} 1-\frac{1}{v} & \frac{-1}{v} & \cdot & \cdot & \cdot & \frac{-1}{v} \\ \frac{-1}{v} & 1-\frac{1}{v} & \cdot & \cdot & \cdot & \frac{-1}{v} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{-1}{v} & \frac{-1}{v} & \cdot & \cdot & \cdot & 1-\frac{1}{v} \end{bmatrix}$$

$$= \frac{1}{H}(I_v - \frac{1}{v} E(v,v))$$

Sufficiency

$V(\hat{t}_j)$ and Cov $(\hat{t}_j, \hat{t}_j1)$ are independent of j and j^1 and hence $V(t_j - t_j1)$ is a constant independent of j and j^1.

In a connected balanced design the roots of C are equal each equal to H.

i.e. $D = HI_{v-1}$

$$\therefore \quad t = \frac{1}{H} NN^{1}Q$$

$$= \frac{1}{H}\left[I_v - \frac{1}{v}E(v,v)\right]Q = \frac{Q}{H}$$

$$U = ND^{-1}N^{1} = \frac{1}{H}NN^{1}$$

$$= \frac{1}{H}\left[I_v - \frac{1}{v}E(v,v)\right]$$

i.e. $UH = (I_v - \frac{1}{v}E(v,v)$

$$\text{Cov}\,(Q_i, Q_j) = H^2\,\text{Cov}\,(t_i, t_j)$$

$$C = H^2\left[\frac{1}{H}\left\{I_v - \frac{1}{v}E(v,v)\right\}\right]$$

$$= H\left\{I_v - \frac{1}{v}E(v,v)\right\}$$

$$= R - N\ K^{-1}N^{1}$$

where N is the incidence matrix of the design

$$\therefore \text{tr}(NK^{-1}N^{1}) = \text{tr}(C+R)$$

$$= \text{tr}\left[R - H(I_v - \frac{1}{v}E(v,v)\right]$$

$$\sum_i \sum_j \frac{n_{ij}^2}{K_j} = \sum ri - H(v-1)$$

$$\therefore \quad H = \frac{ri - \sum_i \sum_j \frac{n_{ij}^2}{K_j}}{v-1}$$

If the design is balanced binary equireplicate

$$H = \frac{vr-b}{v-1} \text{ for, } \sum_i \sum_j \frac{n_{ij}^2}{K_j} = \sum_j \sum_i \frac{n_{ij}^2}{K_j} = b.$$

Now

$$NK^{-1}N^1 = R - H(I_v - \frac{1}{v} E(v,v))$$

$$r - H(1 - \frac{1}{v}) = r - \frac{v-1}{v} \left(\frac{vr-b}{v-1}\right)$$

$$= \frac{vr - rv + b}{v}$$

$$= \frac{b}{v}$$

$$\therefore \quad \frac{H}{v} = \frac{vr-b}{v(v-1)}$$

$$NK^{-1}N' = \begin{bmatrix} \frac{b}{v} & \frac{vr-b}{v(v-1)} & \cdots & \frac{vr-b}{v(v-1)} \\ \frac{vr-b}{v(v-1)} & \frac{b}{v} & \cdots & \frac{vr-b}{v(v-1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{vr-b}{v(v-1)} & \frac{vr-b}{v(v-1)} & \cdots & \frac{b}{v} \end{bmatrix}$$

$$|NK^{-1}N^1| = \left[\frac{b}{v} + (v-1)\frac{vr-b}{v(v-1)}\right]\left[\frac{b}{v} - \frac{vr-b}{v(v-1)}\right]^{v-1}$$

$$= r(b-r)^{v-1}$$

$$> 0 \text{ and } b > r$$

Hence $b \geq v$

Thus for a connected binary equireplicate balanced design b cannot be less than v.

One method generally employed in construction of designs is to deduce them from existing ones. In similar lines attempts have been made by many authors to construct balanced n-ary designs from the existing BIB designs. Nigam et al. (1977) obtained balanced n-ary designs from BIB designs. The substance of their approach is as follows.

Assume that there are u BIB designs each in v treatments $t_1, t_2, \ldots, t_v$. Let the parameters of the i^{th} design be $(v, b_i, r_i, k_i, \lambda_i)$. Augment each of the b_1 blocks of the first design by $K_1{}^1$ plots, each of the b_2 blocks of the second by $K_2{}^1$ plots etc. each of the b_{u-1} blocks of the $(u-1)^{th}$ design by K_{u-1} plots, and apply a new treatment t_0 into all these plots. Repeat each of the augmented blocks m times. The u^{th} design is then repeated p times such that the design obtained is balanced. If maximum value of $K_j{}^1$ $(j=1, \ldots, u-1)$ is $n-1$ the design obtained is n-ary. Since the design should be balanced the C matrix should have all its diagonal elements equal and all off diagonal elements equal. This property can be used to determine the value of p in terms of the parameters of the original designs and $K_1^1, \ldots, K_{u-1}^1$. It is easy to show that

$$p/m = (K_u / \lambda_u) \left[\sum_{j=1}^{u-1} (K_j^1 r_j - \lambda_j)/K_j + K_j^1) \right]$$

The above procedure for the construction of balanced n-ary designs is a generalisation of the method by Kulshrestha et al. (1972) for obtaining some designs with two block sizes.

The design would require too many experimental units. In such cases nearly balanced designs, which may serve as a substitute whenever a suitable design is not available for a given number of treatments, have also been suggested.

Construction of non-proper designs from BIB designs was discussed by John (1964). In these designs the replications can be unequal. There were two types of blocks. One type had size K^1+1 containing a new treatment t_0, K^1 times and one treatment $t_j (j=1, 2, \ldots, v)$ of the BIBD. The other set of blocks consisted of the blocks of the BIBD each having size K. Evidently $\frac{K^1}{K^1+1} = \frac{\lambda}{K}$.

The design can be constructed if the corresponding BIBD exists. It was shown by the author that when $K^1=3$, $K=3$

such designs exist. The blocks of one design for v=4 are (001), (002), (003), (004), (123), (124), (134), (234).

For v=4 the following BIBD exists

blocks

1	1	1	1	0
2	1	1	0	1
3	1	0	1	1
4	0	1	1	1

Adding the blocks (001), (002), (003), (004) to the above design will give a balanced ternary design in 5 treatments in blocks of size 3.

A slightly different method was adopted by Nigam (1974) for obtaining balanced, equireplicate, proper ternary designs. This was done by addition of blocks of balanced incomplete block designs having equal number of treatments. Let N_1 and N_2 be two balanced incomplete block designs with parameters ($v, b_1, r_1, k_1, \lambda_1$ and $v, b_2, r_2, k_2, \lambda_2$ respectively. Now add to the i^{th} block of the first design to the j^{th} block of the second design for $i=1, 2, \ldots, b_1$) $j=1, 2, \ldots, b_2$. The b_1b_2 blocks obtained each of size K_1+K_2 will form a balanced ternary design. Therefore from any given BIBD it is possible to construct a ternary design. It was evident from this result that whenever a t-ary balanced design exists, by resorting to the addition of blocks indicated above a (2t-1)-ary balanced design can be constructed. It can be illustrated as follows. The incidence matrix of a BIBD having v=4, K=2, r=3, b=r, λ=1 is

$$N = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

A BIBD with v=4, r=1, k=1, λ =0 and b=4 is an identity matrix of order 4. Adding each column of this matrix to the above incidence matrix a balanced design having 24 blocks each having size 3 is obtained.

The author has further shown that for any given BIB design adding its i^{th} column to the j^{th} column $i \leq j$ a more desirable balanced ternary design with smaller number of blocks can be obtained.

In the designs constructed by Nigam (1974) the number of blocks and block size were quite large. Following Nigam, Tyagi and Rizvi (1979) suggested some modifications so as to reduce the number of blocks and block size of the ternary (or n-ary) designs. In their approach a BIB design with parameters v, b, r, k, λ whose bxv incidence matrix is N and another BIB design with v treatments and $N^* = nI_v$ as incidence matrix, n is a positive integer was considered. Now following the same procedure given by Nigam blocks were constructed.

It was further shown that the number of blocks and block size could be reduced by adding the elements of the j^{th} block by N^* to only those columns of N which contain unity in the j^{th} column.

The procedure can be illustrated as follows:

Consider the case of a 4 treatment design associated to the BIB design with parameters v=4, b=6, r=3, k=2, λ=1 for which the incidence matrix is

$$N = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

Now if the first column of the identity matrix are added to those of first, second and third rows of N; the elements of second column of I_4 are added to those of first, fourth and fifth columns of N and so on we get the following incidence matrix in 12 blocks.

$$N = \begin{bmatrix} 2 & 2 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \end{bmatrix}$$

Nigam has also developed ternary design for 4 treatments in 12 blocks, but the block size was 4 whereas in the above design the block size is 3.

Proper equireplicate balanced n-ary designs were the interest of Murthy and Das (1967) and Surendran and Sunny (1979). The logic of the designs obtained by the former can be is in fact same as the method of constructing orthogonal arrays. To explain it we give a simple case. The two orthogonal latin squares of side 3 are

$$\begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix}$$

Write the rows and columns of the first as columns of a new array. To this add the columns obtained by taking the numbers corresponding to the same number of the second latin square when it is superimposed on the first. The array obtained will be

0	2	1	0	1	2	0	1	2
1	0	2	2	0	1	1	2	0
2	1	0	1	2	0	2	0	1

This is an orthogonal array. The rows are such that between any two of them the vector $\binom{p}{q}$ will occur exactly the same number of times where no distinction is made between $\binom{p}{q}$ and $\binom{q}{p}$. Therefore it follows that the above array is balanced n-ary design where n=3 provided the columns are taken as blocks. If the numbers 0, 1, 2 are replaced by positive integers p_0, p_1 and p_2, still it will be a n-ary design. The above construction will hold true so long as orthogonal latin squares can be found. Hence from SxS orthogonal latin squares we can construct a number of n-ary designs by replacing the numbers 0, 1, 2, ... (s-1) by appropriate +ve integers.

That Murthy and Das were making use of associable balanced incomplete block designs was brought out by Surendran and Sunny (1979). Two BIB designs with incidence matrices N_1 and N_2 both of order bxv are associable if $N_1 + N_2$ = zero-one matrix and $N_1 N_2{}^1 = A\,[E(v,v)-I]$.

Suppose we group the elements of a orthogonal array into different groups. If the elements of one group are replaced by one and those of the rest by zero and this is done with respect to all groups we will get associable designs. This method is true even if a single integer of the orthogonal array is replaced by one and the rest by zero. It was these associable designs, which were used by Das and Murthy. They were added after multiplying each by an appropriate number to get balanced n-ary design. Generalising this concept Sunny and Surendran (1979) showed that, if N_i, i=1, 2, ... , K are nonnegative integers, then $N=P_1N_1 + ,.....+P_kN_k$ is a proper balanced n-ary design, if largest among the P_i's is (n-1).

MATERIALS AND METHODS

MATERIALS AND METHODS

Several methods are in existence for the construction of designs. One broad approach is to make use of existing designs to generate new ones. The methods generally employed are:

(1) inversion or dualisation

(2) block section

(3) block intersection

(4) complementation, and

(5) Kronecker product.

In this thesis these are employed to construct n-ary balanced designs. Apart from these certain other unique procedures are also made use of to generate them. We shall describe the above methods one by one in relation to BIB designs.

If N is the incidence matrix of the given design its transpose N' will give the inverted design. It is also known as the dual of N.

In the case of symmetrical BIBD it can be shown that the dual design will be same as the original design. This will be true if

$$NN' = N'N$$

as both N and N' are binary. If N is a BIBD with parameters $b=v$, $r=k$, λ

$$NN' = \begin{bmatrix} r & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & r & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot & r \end{bmatrix}$$

$$= (r-\lambda)\, I(v) + \lambda E(v,v)$$

and

$$\left| NN' \right| = rk(r-\lambda)^{v-1}$$

The cofactor of the element in the leading diagonal of NN' is

$$\begin{vmatrix} r & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & r & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot & r \end{vmatrix}_{(v-1)x(v-1)}$$

$$= (r+(v-2)\lambda)\,(r-\lambda)^{v-2}$$

Cofactor of an element other than r is

$$\begin{vmatrix} \lambda & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & r & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot & r \end{vmatrix}$$

$$= -\lambda(r-\lambda)^{v-2}$$

Also $r+(v-2)\lambda = r+(v-1)\lambda - \lambda$

$= r+r(r-1) - \lambda$

$= r^2 - \lambda$

for any BIBD $\lambda(v-1) = r(k-1)$

Therefore

$$(NN')^{-1} = \begin{vmatrix} r^2-\lambda & -\lambda & \cdot & \cdot & \cdot & -\lambda \\ -\lambda & r^2-\lambda & \cdot & \cdot & \cdot & -\lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\lambda & -\lambda & \cdot & \cdot & \cdot & r^2-\lambda \end{vmatrix} \frac{(r-\lambda)^{v-1}}{r^2(r-\lambda)^{v-1}}.$$

$$= \frac{1}{r^2(r-\lambda)} \left[r^2 I(v) - \lambda E(v,v)\right]$$

Hence $$N'^{-1}N^{-1} = \frac{1}{r^2(r-\lambda)} \left[r^2 I(v) - \lambda E(v,v)\right]$$

Premultiplying this relation by N' and post multiplying by N we get

$$I(v) = \frac{1}{r^2(r-\lambda)} \left[r^2 N'N - \lambda r^2 E(v,v)\right]$$

$$= \frac{1}{(r-\lambda)} \left[N'N - \lambda E(v,v)\right]$$

i.e. $N'N = (r-\lambda)I(v) + \lambda E(v,v) = NN'$;

thus showing that N' is the same design as N.

The method of complementation can be described as follows. Let there be a BIBD with parameters v, b, r, k, λ. Construct a new design by taking as its i^{th} block all treatments not contained in the i^{th} block of BIBD. This is called complementary design of the given BIBD.

Symbolically the design complementary to N is

$$N^* = E(v,b)-N$$

$$N^*N^{*1} = [E(v,b)-N)] \; [E(v,b)-N]^1$$
$$= [E(v,b)-N] \; [E(b,v)-N^1]$$
$$= [E(v,b) \; E(b,v)]-NE(b,v)-E(v,b)N^1+NN^1$$
$$= (r-\lambda)I+(b-2r+\lambda)E(v,v)$$

since N^* has block size $v-k$, number of blocks b and number of treatments v, it is a BIBD with parameters $v'=v$, $b'=b$, $r'=b-r$, $k'=v-k$, $\lambda'=b-2r+\lambda$.

In block section a symmetrical BIBD with parameters v, b, r, k, λ is considered and a new design is constructed from this as follows. Take any arbitrary block. Since the relative position of a block will not affect BIBD, we shall take the first block. Drop this block and omit the treatments in this block from the remaining $(b-1)$ blocks. Then we get a BIBD. Since we drop the k treatments in the first block the number of treatments in the new design is $(v-k)$. As one block is dropped the number of blocks of the design will be $(b-1)$.

As every block of the symmetrical BIBD has λ treatments common with each of the other blocks the new design has block size $k-\lambda$. Evidently replication of the treatments is r and the value of r is unaffected. Hence the new design is a BIBD with $v'=v-k$, $b'=b-1$, $K'=k-\lambda$, $r'=r$, $\lambda'=\lambda$.

A BIBD can be constructed from a symmetrical BIBD by dropping an arbitrary block say the first block, but retaining in the remaining blocks only those treatments which were in the first block that was dropped. This procedure is called block intersection. The parameters of this new design are $v'=k$, $b'=b-1$, $r'=r-1$, $k'=\lambda$, $\lambda'=\lambda-1$.

Every treatment of the first block occur once in that block and hence will occur $(r-1)$ times in the remaining $(b-1)$ blocks. Every pair of treatments of the first block occur once in that block. Hence each of these pairs occur together in $(\lambda-1)$ blocks in the remaining $(b-1)$ blocks. As λ is the number of treatments common between any two blocks of a symmetrical BIBD, every block of the generated design will be of size λ. Block intersection yields a useful design only when λ is at least 2.

We shall now introduce Kronecker product.

Definition 3.1. If $A=(a_{ij})$ is an $m \times n$ matrix and $B=(b_{ij})$ is an $p \times q$ matrix, the Kronecker product (direct product) of the matrices A and B, denoted by $A \times B$ is an $np \times mq$ matrix.

The concept of Kronecker product is also very useful in construction of designs. A Kronecker product of designs was defined by Vartak (1960) as follows:

Definition 3.2. If N_1 is the incidence matrix of a design D_1 and N_2 is the incidence matrix of another design D_2, the design with incidence matrix $N_1 x N_2$, where x is Kronecker product of matrices, is called Kronecker product of Designs D_1 and D_2.

The meaning of this definition is as follows:
Let D_i be the design in V_i treatments arranged in b_i blocks of size k_i, each treatment being replicated r_i times (i=1,2). The V_1V_2 treatments in the Kronecker product of designs are the ordered pair of treatments (α, β), α belonging to D_1 and β belonging to D_2. The b_1b_2 blocks are formed by taking any block of D_1 and forming ordered pairs of these treatments with the treatments occurring in any block D_2. For example, D_1 and D_2 are

D_1	D_2
(1,2)	(1,2,4,5)
(1,3)	(3,4,1,2)
(2,3)	(5,2,3,1)

then the Kronecker product of designs D_1 and D_2 is

(1,1), (1,2), (1,4), (1,5), (2,1), (2,2), (2,4), (2,5)
(1,3), (1,4), (1,1), (1,2), (2,3), (2,4), (2,1), (2,2)
(1,5), (1,2), (1,3), (1,1), (2,5), (2,2), (2,3), (2,1)
(1,1), (1,2), (1,4), (1,5), (3,1), (3,2), (3,4), (3,5)
(1,3), (1,4), (1,1), (1,2), (3,3), (3,4), (3,1), (3,2)
(1,5), (1,2), (1,3), (1,1), (3,5), (3,2), (3,3), (3,1)
(2,1), (2,2), (2,4), (2,5), (3,1), (3,2), (3,4), (3,5)
(2,3), (2,4), (2,1), (2,2), (3,3), (3,4), (3,1), (3,2)
(2,5), (2,2), (2,3), (2,1), (3,5), (3,2), (3,3), (3,1).

The following theorem due to Vartak (1960) shows the structure of sets (also known as block structure) of the Kronecker product of designs.

Theorem 3.1. If in a design D_1 there exists a pair of blocks with m_1 treatments in common, then in their Kronecker product there exists a pair of blocks with m_1m_2 treatments in common.

Proof: Let N_1, N_2 and N be the incidence matrices of D_1, D_2 and their Kronecker product, respectively.
Then

$$N = N_1 \times N_2$$

and hence

$$N'N = (N_1'N_1) \times (N_2'N_2)$$

From the statement of the theorem m_1 is an element of $N_1'N_1$ and m_2 is an element of $N_2'N_2$, which implies that there is a pair of sets with m_1m_2 symbols in common.

RESULTS

RESULTS

Eventhough the n-ary designs were introduced by Tocher some 30 years ago no attempt has been made to study the properties of the parameters associated with the design. A study of the relation between parameters may lead to new designs from the existing ones.

4.1. Relationship between parameters.

In balanced incomplete block design we have the result that $r > \lambda$. We shall first suggest an alternative procedure to prove this result.

By Schwart's inequality

(4.1) $(\sum a_i^2)\ (\sum b_i^2) \geq (\sum a_i b_i)^2$

and the equality can arise only if a_i is proportional to b_i. In a BIBD with parameters v, b, r, k, λ

$$\sum_{j=1}^{b} n_{ij} n_{pj} = \lambda$$

and $$\sum_{j=1}^{b} n_{ij}^2 = r$$

Now $$(\sum n_{ij}^2)\ (\sum n_{pj}^2) \geq (\sum n_{ij} n_{pj})^2$$

The equality can arise only if n_{ij} is proportional to n_{pj}. Since each of these two can take only 0, 1 values they can be proportional if every treatment occurs in each block and hence it follows that $r > \lambda$ in a BIBD.

Let us consider proper balanced equireplicate designs. Assume as before that n_{ij} is the number of times the i^{th} treatment occurs in the j^{th} block.

Then

$$NN^1 = \left(\sum_{j=1}^{b} n_{ij} n_{pj}\right)$$

Since the design is balanced as also equireplicate and since the C matrix of the design is

$$(4.2)\quad R - NK^{-1}N^1 = \text{Diag}\,(r, \ldots, r) - \left(\sum_{j=1}^{b} n_{ij} n_{pj}\right)$$

all the diagonal elements of the C matrix should be equal so also should be the off-diagonal elements.

Therefore it follows that

$$(4.3)\quad \sum_{j=1}^{b} n_{ij}^2 = h$$

is a constant for all i and

$$(4.4)\quad \sum_{j=1}^{b} n_{ij} n_{pj} = \lambda \text{ is a constant for any } i \text{ and } p \; (i \neq p)$$

Hence we come to the result that

$$NN^1 = \begin{bmatrix} h & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & h & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot & h \end{bmatrix}$$

We shall now prove that $h > \lambda$

From Schwart's inequality

$$(\sum n_{ij}^2)\ (\sum n_{pj}^2) \geqslant (\sum n_{ij} n_{pj})^2$$

and the equality can hold good if n_{ij} is proportional to n_{pj}. That is every treatment occurs in the same proportion in every block which is impossible and hence $h > \lambda$.

Theorem 4.1. In a proper equireplicate balanced n-ary design $b \geqslant v$.

$$NN^1 = (h-\lambda)\ I(v) + \lambda E(v,v)$$

$$\therefore\ \left|NN^1\right| = \begin{vmatrix} h & \lambda & \cdots & \lambda \\ \lambda & h & \cdots & \lambda \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \lambda & \lambda & \cdots & h \end{vmatrix}$$

$$= \left[h+(v-1)\lambda\right] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda & h & \cdots & \lambda \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \lambda & \lambda & \cdots & h \end{vmatrix}$$

Adding all rows to the first and taking the common factor out.

$$= \left[h+(v-1)\lambda\right] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & h-\lambda & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & h-\lambda \end{vmatrix}$$

Substracting λ times first row from the remaining rows

$$= \left[h+(v-1)\lambda\right] (h-\lambda)^{v-1}$$

This is different from zero as $h > \lambda$

Hence $b \geq v$

Result 4.1. $rk = h+(v-1)\lambda$ in a proper equireplicate balanced n-ary incomplete block design.

Proof. The C-matrix of the design by the notation used above is

$$\text{diag}\,(r, r, \ldots, r) - \begin{bmatrix} \frac{h}{k} & \frac{\lambda}{k} & \cdot & \cdot & \cdot & \frac{\lambda}{k} \\ \lambda/k & h/k & \cdot & \cdot & \cdot & \lambda/k \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda/k & \lambda/k & \cdot & \cdot & \cdot & h/k \end{bmatrix}$$

Since every row of the C-matrix adds to zero we get

$$r - \frac{h}{k} - \frac{\lambda(v-1)}{k} = 0$$

(4.4) i.e. $rk = h+(v-1)\lambda$

i.e. $(rk-h) = (v-1)\lambda$

We define a proper equireplicate symmetrical n-ary balanced design as one having $b=v$.

Theorem 4.2. The dual of a symmetrical proper equireplicate balanced n-ary design is itself.

Proof. We have already shown that in a symmetrical proper equireplicate balanced n-ary design.

$$NN^1 = (h-\lambda)I(v) + \lambda E(v,v)$$

Now the determinant of NN^1, on remembering relation (4.4), is $r^2(h-\lambda)^{v-1}$.

The cofactor of h in NN^1 is

$$\begin{vmatrix} h & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & h & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot & h \end{vmatrix}_{(v-1)(v-1)}$$

Adding all rows to the first and taking the common factor out we get the value of cofactor as

$$\left[h+(v-2)\lambda\right] \begin{vmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \lambda & h & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot & h \end{vmatrix}$$

$$= \left[h+(v-2)\lambda\right](h-\lambda)^{v-2}$$

on simplification.

Remembering (4.4) this simplifies to

(4.5) $(r^2-\lambda)\,(h-\lambda)^{v-2}$

Cofactor of λ in NN^1 is

$$\begin{vmatrix} \lambda & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & h & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot & h \end{vmatrix}_{(v-1)(v-1)}$$

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Which simplifies to $-\lambda(h-\lambda)^{v-2}$

$$\therefore \quad (NN^1)^{-1} = \frac{(h-\lambda)^{v-2}}{r^2(h-\lambda)^{v-1}} \begin{bmatrix} r^2-\lambda & -\lambda & \cdot & \cdot & \cdot & -\lambda \\ -\lambda & r^2-\lambda & \cdot & \cdot & \cdot & -\lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\lambda & -\lambda & \cdot & \cdot & \cdot & r^2-\lambda \end{bmatrix}$$

$$= \frac{1}{r^2(h-\lambda)} \left[r^2 I(v) - \lambda E(v,v) \right]$$

Since b=v and $|NN^1|$ is different from zero N is a non-singular matrix and is therefore inversible. Hence

$$(N^1)^{-1}N^{-1} = \frac{1}{r^2(h-\lambda)} \left[r^2 I(v) - \lambda E(v,v) \right]$$

Premultiplity this relation by N^1 and post multiply by N. Then remembering that sum of any row or column of N is r we get

$$I = \frac{1}{r^2(h-\lambda)} \left[r^2 N^1 N - \lambda r^2 E(v,v) \right]$$

$$= \frac{1}{(h-\lambda)} \left[N^1 N - \lambda E(v,v) \right]$$

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$$\text{or } N^1 N = (h-\lambda) I(v) + \lambda E(v,v) = NN^1$$

This proves the result.

Therefore if we invert a symmetrical n-ary proper equi-replicate design no new design will be obtained. As an illustration we quote a symmetrical design (D) given by Tocher (1952) wherein rows represent the replications of the treatments and columns the blocks

$$\begin{matrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 \end{matrix}$$

Further we have in a symmetrical proper equireplicate balanced n-ary design.

$$|NN^1| = |N|^2 = r^2(h-r)^{v-1}$$

Theorem 4.3. If n_{ij} denotes the number of times the i^{th} treatment occurs in the j^{th} block of a proper n-ary balanced design it will be equireplicate if $\sum_{j=1}^{b} n_{ij}^2$ = a constant.

If $h = \sum_{j=1}^{b} n_{ij}^2$ it is easy to show that

$$V(Qi) = (r_i - \frac{h}{k})\sigma^2$$

Since this is to be independent of i for the design to be balanced it follows that $r_1 = r_2 = \ldots = r_v$

Theorem 4.4. In any balanced equireplicate n-ary design $b \geq v$.

Proof. Let r be the number of replications of each treatment and k_j be the size of the j^{th} block.

Then

$$C = R - N K^{-1} N^1$$

(4.6) $= \text{diag}\,(r, \ldots, r) - \sum_{j=1}^{b} \frac{n_{ij} n_{mj}}{k_j}$

$i = 1, 2, \ldots, v$

$j = 1, 2, \ldots, b$

If L is the matrix of eigen vectors of C the first column of L is $\frac{1}{\sqrt{v}}\, E(v{\times}1)$ and

$$L^1 CL = L^1 (R - N\,K^{-1} N^1)\,L$$

$$= R - (L^1 (NK^{-1} N^1)\,L$$

i.e. diag. $(0, g, \ldots, g) = R - L^1 NK^{-1} N^1 L$

(4.7) i.e. $L^1 NK^{-1} N^1 L = \text{diag}\,(r, r-g, \ldots, r-g)$

This relation shows that $NK^{-1}N^1$ will have all diagonal elements equal and all off diagonal elements equal.

In (4.7) if $r \neq g$ the rank of $NK^{-1}N^1$ will be v.

Let if possible $r = g$

Then $NK^{-1}N^1 = \frac{r}{v} E(v, v)$

Comparing this with (4.6) such a situation can arise only if $\sum_{j=1}^{b} \frac{n_{ij}^2}{k_j} = \sum_{j=1}^{b} \frac{n_{ij} n_{mj}}{k_j}$

i.e. the treatments occur proportionately in all blocks which is impossible. Hence $g \neq r$ and therefore the rank of $NK^{-1}N^1$ is v.

i.e. $b \geq v$

One method of construction of balanced incomplete block design is by block section and block intersection applied to symmetrical BIBD's. However the same procedure cannot be extended to n-ary designs. The block section applied to symmetrical proper equireplicate balanced n-ary designs could give a design N such that NN^1 is of the form

$$(h-\lambda)I(v)+\lambda E(v,v)$$

However the block size of the design will not be a constant and therefore the procedure does not give the desired design. As an illustration if we drop the first block of the symmetrical ternary design by Tocher (1952) given earlier in this chapter we get the design

$$N = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix}$$

$$NN^1 = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

each treatment is replicated 4 times and block sizes are 3, 3, 2, 2, 2.

The C matrix of the design is

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 7/3 & 1 & 2/3 \\ 1 & 7/3 & 2/3 \\ 2/3 & 2/3 & 8/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 & -1 & -2/3 \\ -1 & 5/3 & -2/3 \\ -2/3 & -2/3 & 4/3 \end{bmatrix}$$

and this shows that the design obtained by block section is not balanced. If we apply block intersection to symmetrical proper equireplicate n-ary designs the procedure may not lead to balanced design.

As an example if we drop the first block of the symmetrical design due to Tocher (1952) and keep only the treatments of this block in the rest of the block we get the design

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}$$

which is clearly not balanced.

Though block section and block intersection are not fruitful in giving n-ary designs the method of complementation usually used in incomplete block designs for the construction of designs is useful in the case of n-ary designs. But the complementation in this case takes a slightly different form.

Theorem 4.5. Let N be a balanced proper n-ary design. Then $N_1 = (n-1)E(v,b)-N$ is a proper balanced n-ary design.

Proof. It is easy to show that sum of every row of N_1 is the same and also column sums of N_1 are equal. Therefore if we can show that N_1N_1' can be thrown into the form $(h-\lambda)I(v)+\lambda E(v,v)$ then N is a proper equireplicate n-ary design. We note that the largest element of N_1 is $(n-1)$ and the smallest element zero as N contains some elements equal to $(n-1)$.

$$\begin{aligned} \text{Then } N_1N_1' &= \left[(n-1)E(v,b)-N\right]\left[(n-1)E(v,b)-N\right]' \\ &= (n-1)^2 bE(v,v)-(n-1)rE(b,v)-(n-1)rE(b,v)+ \\ &\quad (h-\lambda)I(v)+\lambda E(v,v) \\ &= E(v,v)\left[(n-1)^2 b-2r(n-1)+\lambda\right]+(h-\lambda)I(v) \end{aligned}$$

Further row sum of N_1 and column sum of the same design are constants. Hence N_1 is a balanced equireplicate proper design.

4.2. Balanced designs with unequal replications.

Consider a balanced incomplete block design in 4 treatments having the following incidence matrix.

$$N = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

To this we add 4 blocks each of size 3 and each consisting of a new treatment 0 and an old treatment replicated twice.

That is to say the composition of the i^{th} block will be $(0, i, i)$ $i = 1, 2, 3, 4$. The incidence matrix of the design is

$$N_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This is a proper design and

$$N_1 K_1^{-1} N_1^{1} = \frac{1}{3} \begin{bmatrix} 4 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 7 \end{bmatrix}$$

$R = \text{daig}\ (4,5,5,5,5,\ \ )$ and hence

$$C = \begin{bmatrix} 8/3 & -2/3 & -2/3 & -2/3 & -2/3 \\ -2/3 & 8/3 & -2/3 & -2/3 & -2/3 \\ -2/3 & -2/3 & 8/3 & -2/3 & -2/3 \\ -2/3 & -2/3 & -2/3 & 8/3 & -2/3 \\ -2/3 & -2/3 & -2/3 & -2/3 & 8/3 \end{bmatrix}$$

which shows the design is balanced eventhough the replications of the treatments are not equal. This design was obtained as a variation of the design given by John (1964).

John's design consist of block (0,0,1) (0,0,2) (0,0,3) (0,0,4) and N. The design given above can be further generalised into the following theorem.

Theorem 4.6. Let N be a BIBD in treatments 1, 2, ... , v with parameters v, b, r, k, λ. Add to this design v blocks the i^{th} one containing a new treatment 0 and the treatment i, the latter repeated (k-1) times in the block. The design N_1 in (v+1) treatments is a balanced k-ary design provided r=(v-1) and λ = k-1.

Proof. $C_{oo} = v - \frac{v}{k} = \frac{v(k-1)}{k}$

$$C_{ii} = r+(k-1)-\frac{(k-1)^2}{k} - \frac{r}{k}$$

$$= \frac{(k-1)}{k} \quad (r+1)$$

as C_{ii} should be equal to C_{oo} for the design to be balanced we get the condition r=(v-1)

Further $C_{oi} = \frac{-(k-1)}{k}$

and $C_{ii^1} = -\frac{\lambda}{k}$

For the design to be balanced the offdiagonal elements should be equal. This leads to the relation $\frac{(k-1)}{k} = \frac{\lambda}{k}$ or λ =k-1. Therefore the above k-ary design can exist only if v, b, (v-1), k, (k-1) exists. We know that a design of the type r=k=(v-1) and λ=(v-2) always exists.

4.3. Balanced designs with unequal block size and unequal replications.

There are situations in which the available animals cannot be used completely for the experiment using conventional designs. Consider for example an experiment in which 7 diets are to be tried on piglets from 3 litters each of size 10. If we use the conventional experiment we can make use of only a randomised block with 3 replications. This will leave out the remaining 9 animals eventhough a replication of 3 may be considered inadequate. In this context let us consider a design of the following type consisting of the treatments 0, 1, 2, . . . 6. The treatments of the blocks are indicated in brackets. (0,1), (0,2), (0,3), (0,4), (0,5), (0.6), (1,2,3,4,5,6), (1,2,3,4,5,6), (1,2,3,4,5,6) Then

$$N_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \text{diag}\,(6,4,4,4,4,4)$$

$$K = (2,2,2,2,2,2,6,6,6)$$

and

$$NK^{-1}N^{1} = \begin{bmatrix} 3 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 3 & -1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 3 & -1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 3 & -1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & -1/2 & 3 & -1/2 \\ -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 3 \end{bmatrix}$$

and the design is balanced. The above indicates an easy but effective procedure for constructing balanced n-ary designs with unequal replications and unequal block sizes for practical purposes. This is actually an extension of the procedure contained in theorem (4.5). Thus the following theorem.

Theorem 4.7. Let there be a randomised block design with v treatments $1, 2, \ldots, v$. Then if N is this design the design N_1 obtained by adding v blocks each of size 2, the i^{th} containing a new treatment and the old treatment i will be balanced binary design for $(v+1)$ treatments with different block size and different replications provided $r = \frac{v}{2}$

Proof $C_{oo} = v - \frac{v}{2}$

$C_{ii} = (r+1) - \frac{1}{2} - \frac{r}{v}$

For balance we must have

$C_{oo} = C_{ii}$

i.e. $v - \frac{v}{2} = (r+1) - \frac{1}{2} - \frac{r}{v}$

i.e. $\frac{v}{r} = 2$ on simplification

$C_{oi} = -\frac{1}{2}$ and $C_{ii}1 = -\frac{r}{v}$

same relation as before. This proves the theorem

4.4 Balanced designs with unequal block sizes and equal replications.

The above idea can be extended. For example, if we associate with a randomised block design of v treatments in r blocks (0,1,1) . . . .(0,v,v) Where 0 is a new treatment and 1,2, ...v are old treatments, the design obtained will be balanced binary for (v+1) treatments if $r = \frac{2}{3}v$. Thus for 6 treatments the design (0,1,1) (0,2,2), (0,3,3), (0,4,4), (0,5,5), (0,6,6) (1,2,3,4,5,6) (1,2,3,4,5,6) (1,2,3,4,5,6) (1,2,3,4,5,6) will be balanced. For this design

R = daig (6,6,6,6,6,6), K = (3,3,3,3,3,3,6,6,6)

$$NK^{-1}N^{1} = \begin{bmatrix} 2 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 \\ 2/3 & 2 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2 & 2/3 & 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 & 2 & 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 & 2/3 & 2 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 & 2/3 & 2/3 & 2 & 2/3 \\ 2/3 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 \\ -2/3 & 4 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 \\ -2/3 & -2/3 & 4 & -2/3 & -2/3 & -2/3 & -2/3 \\ -2/3 & -2/3 & -2/3 & 4 & -2/3 & -2/3 & -2/3 \\ -2/3 & -2/3 & -2/3 & -2/3 & 4 & -2/3 & -2/3 \\ -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & 4 & -2/3 \\ -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & 4 \end{bmatrix}$$

4.5. Kronecker product and balanced n-ary designs.

We shall now use the method of Kronecker product to the construction of proper equireplicate balanced n-ary designs. The method is contained in the theorems given below.

Theorem 4.8. Let N_1 and N_2 be two BIB designs with parameters $v, b_1, r_1, k_1, \lambda_1$ and $v, b_2, r_2, k_2, \lambda_2$ respectively. For positive integral values of a_1 and a_2, $a_1 E(1,b_2) x N_1 + a_2 N_2 \times E(1,b_1)$ is in general a proper equireplicate n-ary design provided $a_1+a_2+1 = n$

Proof. Let $N = a_1E(1,b_2)xN_1+a_2N_2 \times E(1,b_1)$.

Then as

$$(A \times B)\ (A \times B)^1 = AA^1 \times BB^1,$$

$$NN^1 = \left[a_1E(1,b_2) \times N_1+a_2N_2 \times E(1,b_1)\right] \left[a_1E(1,b_2)^1 \times N_1^1+a_2N_2^1 \times E(1,b_1)\right]^1$$

$$= a_1^2E(1,b_2)E(1,b_2)^1 \times N_1N_1^1 +$$

$$a_2^2\ N_2N_2^1 \times E(1,b_1)E(1,b_1)^1 +$$

$$2a_1a_2\ E(1,b_2)\ N_2^1 \times N_1E(1,b_1)^1$$

(4.8) $\therefore\ NN^1 = a_1^2b_2N_1N_1^1 + a_2^2N_2N_2^1b_1 + 2va_1a_2r_1r_2E(v,v)$

$$= a_1^2b_2 \begin{bmatrix} r_1 & \lambda_1 & \cdot & \cdot & \cdot & \lambda_1 \\ \lambda_1 & r_1 & \cdot & \cdot & \cdot & \lambda_1 \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \lambda_1 & \lambda_1 & \cdot & \cdot & & r_1 \end{bmatrix} + a_2^2b_1 \begin{bmatrix} r_2 & \lambda_2 & \cdot & \cdot & \cdot & \lambda_2 \\ \lambda_2 & r_2 & \cdot & \cdot & \cdot & \lambda_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_2 & \lambda_2 & \cdot & \cdot & \cdot & r_2 \end{bmatrix}$$

$$+ 2va_1a_2r_1r_2 \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

(4.9) i.e. $NN^1 = \begin{bmatrix} h & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & h & \cdot & \cdot & \cdot & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdot & \cdot & \cdot\cdot & h \end{bmatrix}$ where

(4.10) $h = a_1^2 b_2 r_1 + a_2^2 b_1 r_2 + 2va_1 a_2 r_1 r_2$

(4.11) $\lambda = a_1^2 b_2 \lambda_1 + a_2^2 b_1 \lambda_2 + 2va_1 a_2 r_1 r_2$

The block size of N is

(4.12) $a_1k_1+a_2k_2$ and the replication of each treatment in N is

(4.13) $a_1b_2r_1+a_2b_1r_1$

The equations (4.12) and (4.13) along with (4.9) shows that N is a proper equireplicate n-ary design where evidently $n-1 = a_1+a_2$.

Corollary 4.1. Taking $a_1=a_2=1$ We see that $E(1,b_2)xN_1+N_2xE(1,b_1) = N$ is a ternary proper equireplicate design with number of blocks b_1b_2, number of treatments v, $r=b_1r_2+b_2r_1$ and $k=k_1+k_2$.

Corollary 4.2. If $a_1=1$ $a_2=n-2$ and $N_2= I(v)$ a n-ary design with $b=b_1v$ $r=r_1b_1$ $k=k_1+1$ and number of treatments equal to v is obtained.

Theorem 4.9. N_1 and N_2 are two balanced proper equireplicate n_1-ary and n_2-ary designs in v treatments with b_1, b_2 blocks respectively. If a_1 and a_2 are positive integers $a_1E(1,b_2) \times N_1+a_2N_2xE(1,b_1)$ is a n-ary balanced equireplicate proper design with b_1b_2 blocks where $n=a_1(n_1-1)+a_2(n_2-1)+1$.

Proof. Since N_1 and N_2 are balanced proper equireplicate designs $N_1N_1^1$ and $N_2N_2^1$ can be thrown into the form

$$(4.14)\quad N_1N_1^1 = (h_1-\lambda_1)\, I(v)+\lambda_1 E(v,v)$$

$$(4.15)\quad N_2N_2^1 = (h_2-\lambda_2)\, I(v)+\lambda_2 E(v,v)$$

Now defining

$$(4.16)\quad N = a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)$$

$$(4.17)\quad NN^1 = \left[a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)\right]\left[a_1E(1,b_2)^1 \times N_1^1+a_2N_2^1xE(1,b_1)\right]$$

$$= a_1^2b_2N_1N_1^1+a_2^2b_1N_2N_2^1+2a_1a_2vr_1r_2E(v,v)$$

Using equation (4.14) and (4.15) in this relation,

$$NN^1 = a_1^2b_2((h_1-\lambda_1)\, I(v)+\lambda_1E(v,v)$$

$$+a_2^2b_1\ \ (h_2-\lambda_2)I(v)+\lambda_2E(v,v)$$

$$+2a_1a_2vr_1r_2E(v,v)$$

$$= (h-\lambda)I(v)+\lambda E(v,v) \text{ where}$$

$$(4.18)\quad h = a_1^2b_2h_1+a_2^2b_1h_2+2a_1a_2vr_1r_2$$

$$(4.19)\quad \lambda = a_1^2b_2\lambda_1+a_2^2b_1\lambda_2+2a_1a_2vr_1r_2$$

Further we note that

$$(4.20)\quad k = a_1k_1+a_2k_2.$$

and

$$(4.21)\quad r = a_1b_2r_1+a_2b_1r_2.$$

where k and r are the block size and replications of treatments in N and k_i, r_i are block size and replications of the treatments in the design N_i, $i = 1,2$.

Corollary 4.3. If $a_1=a_2=1$, $N_1=N_2$, a t-ary balanced equireplicate proper design then N is a proper equireplicate balanced $(2t-1)$-ary design.

Corollary 4.4. If $a_1=a_2=1$, N_1 a p-ary design and N_2 a s-ary design then N is a $p+s-1$-ary design.

Example 4.1. Taking N_1 equal to the design D on page (7) and $N_2=I_6$ we get the following 4-ary design with $v=6$, $b=36$ $r=30$, $k=5$ and $\lambda=20$.

3	2	2	1	1	1	2	1	1	0	0	0	2	1	1	0	0	0
0	2	0	1	0	1	1	3	1	2	1	2	0	2	0	1	0	1
0	0	2	1	0	1	0	0	2	1	0	1	1	1	3	2	1	2
1	0	0	2	1	0	1	0	0	2	1	0	1	0	0	2	1	0
0	1	1	0	2	0	0	1	1	0	2	0	0	1	1	0	2	0
1	0	0	0	1	2	1	0	0	0	1	2	1	0	0	0	1	2

2	1	1	0	0	0	2	1	1	0	0	0	2	1	1	0	0	0
0	2	0	1	0	1	0	2	0	1	0	1	0	2	0	1	0	1
0	0	2	1	0	1	0	0	2	1	0	1	0	0	2	1	0	1
2	1	1	3	2	11	1	0	0	2	1	0	1	0	0	2	1	0
0	1	1	0	2	0	1	2	2	1	3	1	0	1	1	0	2	0
1	0	0	0	1	2	1	0	0	0	1	2	2	1	1	1	2	3

4.6. Nearly balanced designs

Consider a randomised block design in which all but one of the r blocks contain all the v treatments and the first block contains only (v-1) treatments, the first being omitted from it. The C-matrix of the design is easily seen to be

$$C = R - NK^{-1}N'$$

where $R = \text{diag}\,(r-1, r, \ldots, r)$

$$NK^{-1}N' = \begin{bmatrix} \frac{r-1}{v} & \frac{r-1}{v} & \cdots & \frac{r-1}{v} \\ \frac{r-1}{v} & \theta + \frac{r-1}{v} & \cdots & \theta + \frac{r-1}{v} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{r-1}{v} & \theta + \frac{r-1}{v} & \cdots & \theta + \frac{r-1}{v} \end{bmatrix} \quad \text{Where } \theta = \frac{1}{v-1}$$

$$C = \begin{bmatrix} (r-1)-(\frac{r-1}{v}) & -(\frac{r-1}{v}) & \cdots & -(\frac{r-1}{v}) \\ -(\frac{r-1}{v}) & r-(\theta + \frac{r-1}{v}) & \cdots & -(\theta + \frac{r-1}{v}) \\ \vdots & \vdots & \cdots & \vdots \\ -(\frac{r-1}{v}) & -(\theta + \frac{r-1}{v}) & \cdots & -(\theta + \frac{r-1}{v}) \end{bmatrix}$$

$$\left| C - \lambda I \right| = \begin{bmatrix} r-1-(\frac{r-1}{v})-\lambda & -(\frac{r-1}{v}) & \cdots & -(\frac{r-1}{v}) \\ -(\frac{r-1}{v}) & r-(\theta+(\frac{r-1}{v})-\lambda & \cdots & -(\theta+(\frac{r-1}{v}) \\ \vdots & \vdots & \cdots & \vdots \\ -(\frac{r-1}{v}) & -(\theta + \frac{r-1}{v}) & \cdots & r-(\theta + \frac{r-1}{v})-\lambda \end{bmatrix}$$

$$= \begin{vmatrix} -\lambda & -\lambda & \cdot & \cdot & \cdot & -\lambda \\ -(\frac{r-1}{v}) & r-\lambda-\theta-(\frac{r-1}{v}) & \cdot & \cdot & \cdot & -(\frac{r-1}{v})-\theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & \\ -(\frac{r-1}{v}) & -(\frac{r-1}{v})-\theta & \cdot & \cdot & \cdot & r-\lambda-\theta-\frac{r-1}{v} \end{vmatrix}$$

Adding all rows to the first row.

$$= -\lambda \begin{vmatrix} 1 & 0 & 0 & \cdot\ \cdot\ \cdot & 0 \\ -(\frac{r-1}{v}) & r-\lambda-\theta & -\theta & \cdot\cdot\cdot\cdot\cdot & -\theta \\ -(\frac{r-1}{v}) & -\theta & r-\lambda-\theta & \cdot\ \cdot & -\theta \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ -(\frac{r-1}{v}) & -\theta & \cdot & \cdot\ \cdot\ \cdot & r-\lambda-\theta \end{vmatrix}$$

Taking λ outside and substracting 1st row from the remaining rows.

$$= -\lambda \begin{vmatrix} r-\lambda-\theta & -\theta & \cdot\ \cdot\ \cdot\ \cdot & -\theta \\ -\theta & r-\lambda-\theta & \cdot\ \cdot\ \cdot\ \cdot & -\theta \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ -\theta & -\theta & & r-\lambda-\theta \end{vmatrix}_{(v-1)(v-1)}$$

$$= \; -\lambda\left[r-\lambda-(v-1)\theta\right]\begin{vmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -\theta & r-\lambda & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\theta & 0 & \cdot & \cdot & \cdot & r-\lambda \end{vmatrix}$$

Adding all rows to the first and taking the common factor out and substracting first row from the remaining rows.

$$= \quad -\lambda(r-\lambda-(v-1)\,\theta)\,(r-\lambda)^{v-2}$$

The characteristic roots are solution of

$$|C-\lambda I| = 0$$

$$\therefore \quad -\lambda(r-\lambda-(v-1)\,\theta)\,(r-\lambda)^{v-2} = 0$$

gives $\lambda_1 = 0$, $\lambda_2 = r-(v-1)\,\theta = (r-1)$, and $\lambda_3 = \ldots = \lambda_v = r$

Hence range of variation of an elementary contrast (Sylvain Ehrenfeld, 1955)

$$\frac{2\sigma^2}{r} \leq V(\hat{t}_i-\hat{t}_j) \leq \frac{2\sigma^2}{v-1}$$

Average variance $= \left(\frac{1}{r-1} + \frac{r-2}{r}\right) \frac{2\sigma^2}{v-1}$

$$= \frac{2\sigma^2}{r} + \frac{2\sigma^2}{r(r-1)(v-1)}$$

Increase in average variance is

$$\left(\frac{1}{r-1} + \frac{r-2}{r}\right) \frac{1}{v-1} - \frac{1}{r} = \frac{1}{r(r-1)(v-1)}$$

$\therefore$ Percentage increase in variance = $\frac{100}{(r-1)(v-1)}$

which is small for moderate values of v and r.

4.7. Analysis. The importance of balanced n-ary design is that the intrablock analysis is pretty easy. Under the usual linear model for two way classification the treatment effects are estimated from the normal equations

$$C\hat{t} = Q$$

Where C = $(a-b)I + bE(v,v)$ and hence

$$Q_i = a\hat{t}_i + \sum_{\substack{j=1 \\ j\neq i}}^{v} \hat{t}_j$$

$$= (a-b)\hat{t}_i$$

So that

$$\hat{t}_i = \frac{Q_i}{(a-b)}$$

and adjusted sum of squares due to treatment effect is equal to $\frac{Q_i^2}{a-b}$

The rest of the procedure for forming the analysis of variance table is same as in a two-way classification. Further the variance of an elementary contrast is $\frac{2\sigma^2}{a-b}$.

DISCUSSION

DISCUSSION

In a balanced incomplete block design with parameters v, b, r, k, λ we have the result $r > \lambda$. An equivalent result in a proper equireplicate balanced n-ary design is $h > \lambda$, where $h = \sum_j n_{ij}^2$, $\lambda = \sum_j n_{ij} n_{pj}$. It was shown, $b \geq v$ in a proper equireplicate balanced n-ary design and this is an extension of the well known Fisher's inequality in the case of BIBD to the n-ary designs. Again in a BIBD we have the relation $rk - r = \lambda(v-1)$. An equivalent relation in n-ary design is established as $rk - h = \lambda(v-1)$.

In the case of symmetrical incomplete block design the dual is itself. It was shown that if N is a proper equireplicate balanced n-ary design its dual is itself. This implies that dualisation will not lead to a new design. In a symmetrical proper equireplicate balanced n-ary design, $|NN^1| = |N|^2 = r^2(h-\lambda)^{v-1}$. If v is even for the existence of such a design $(h-\lambda)$ should be a perfect square. This is a result comparable with the corresponding result for the existence of a symmetrical BIBD that $(r-\lambda)$ should be perfect square if v is even.

It has been proved by Rao (1958) that a proper balanced binary design is always equireplicate. The result cannot be straightaway extended to the n-ary designs.

However it will be true if $\sum_j n_{ij}^2$ is a constant for all i.

It was shown that in any balanced equireplicate n-ary design $b \geq v$. This is a generalisation of the result of Atiqullah (1961) that in any equireplicate balanced binary design $b \geq v$ which itself is a generalisation of Fisher's inequality with reference to a BIBD. Thus the result $b \geq v$ established in this thesis is the most general extension of the results due to the above authors.

Block section and Block intersection may lead to new designs. However it was shown that this procedure applied to proper equireplicate symmetrical n-ary designs may not lead to new designs. But it was established that the complimentation of a special type leads to new designs which are proper equireplicate n-ary designs.

It was observed by Calinski (1971) that when comparing new varieties the seeds of which are in short supply equal replication may not be possible eventhough equal information on the varieties may be required. Such situations call for construction of balanced designs with unequal replications. The example (011), (022), (033), (044), (123), (124), (134), (234) was obtained as a variation of the design given by John (1964). The idea behind this approach was further generalised into the following theorem.

Let N be a BIBD in treatments 1,2, ...,v with parameters v, b, r, k, λ. Add to this design v blocks the i^{th} one containing a new treatment zero and the treatment i, the latter repeated (k-1) times in the block. The new design N_1 in (v+1) treatments is a balanced k-ary design provided r=(v-1) and λ=k-1.

There are situations in which the available animals cannot be used completely for experiment using conventional designs. These situations may demand designs with unequal replications and unequal block sizes. An experiment involving 7 diets may utilise only 21 animal from three litters each of size 10, if the conventional RBD is used. This will leave out the remaining 9 animals eventhough a replication of 3 may be considered inadequate. Design suited for such occasion are suggested in the following theorem.

Let there be a randomised block design with v treatments 1, 2, ...,v. Then if N is this design the design N_1 obtained by adding v blocks each of size 2, the i^{th} containing a new treatment and the old treatment i will be balanced binary design for (v+1) treatments with unequal block size and unequal replications provided r = $\frac{v}{2}$.

Method of Kronecker product for the construction of designs was formally introduced by Vartak (1955). He depended upon enumeration for the establishment of results. The mathematical approach in this connection was supplied by Surendran (1968). The Kronecker product for the construction of balanced n-ary designs is formally introduced to the literature in the following premier theorem.

Let N_1 and N_2 be two BIB designs with parameters $v, b_1, r_1, k_1, \lambda_1$ and $v, b_2, r_2, k_2, \lambda_2$ respectively. For positive integral values of a_1 and a_2, $a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)$ is in general a proper equireplicate n-ary design provided $a_1+a_2+1 = n$.

Taking $a_1=a_2=1$ we see that $E(1,b_2)xN_1+N_2xE(1,b_1)=N$ is a ternary proper equireplicate design with number of blocks b_1b_2, number of treatments v, $r=b_1r_2+b_2r_1$ and $k=k_1+k_2$. This result was established by Nigam (1974) by the method of enumeration.

If $a_1=1$, $a_2=n-2$ and $N_2=I(v)$ a n-ary design with $b=b_1v$, $r=r_1v+b_1$, $k=k_1+1$ and number of treatments equal to v is obtained. This was proved by Tyagi and Riswi (1979) by a different approach.

Further generalisation of the application of the application of Kronecker product for the construction of n-ary designs is contained in the theorem given below.

N_1 and N_2 are two balanced proper equireplicate n_1-ary and n_2-ary designs in v treatments with b_1, b_2 blocks respectively. If a_1 and a_2 are positive integers $a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)$ is a n-ary balanced equireplicate proper design with b_1b_2 blocks where $n+1=a_1(n_1-1)+a_2(n_2-1)$.

If $a_1=a_2=1$, $N_1=N_2$, a t-ary balanced equireplicate proper design then N is a proper equireplicate (2t-1)-ary design.

This result was first established by Nigam (1974).

If $a_1=a_2=1$, N_1 a p-ary design and N_2 a s-ary design then N is a p+s-1-ary design.

It was observed by Nigam (1974) that if proper equireplicate balanced p-ary and s-ary designs are given in the same number of treatments it is possible to construct p+s-1-ary proper equireplicate balanced design.

The above theorem contains the most general form of constructing equireplicate proper balanced n-ary designs from existing designs of the same type. The results obtained by the previous authors (Nigam, Tyagi and Rizwi) are all particular cases of this theorem. The theorem has added importance as it is based on a sound mathematical procedure and therefore contains the possibilities of further development. It is to be remembered that the previous authors have established results which are particular cases of the theorem by enumeration.

SUMMARY

SUMMARY

An attempt was made to study the properties of the n-ary designs and some relation between the parameters of the design were established. If $h = \sum_j n_{ij}^2$, $\lambda = \sum_j n_{ij} n_{pj}$ in a proper equireplicate balanced n-ary design it is shown that

(i) $h > \lambda$

(ii) $b \geq v$

(iii) $rk = h + (v-1)\lambda$

If N is a proper equireplicate balanced n-ary design it is proved that its dual is itself. Further it was proved that a symmetrical proper equireplicate balanced n-ary design for an even value of v cannot exist if $h-\lambda$ is not a perfect square.

One has to be cautious in extending block section and block inter-section to generate balanced designs from symmetrical proper equireplicate balanced n-ary designs as these procedures may not yield them. However a modified form of complementation was shown to lead to proper equireplicate balanced designs.

In situations like comparison of new varieties of seeds of which are in short supply, equal replication may not be possible. In such situations we have shown that it is possible to construct balanced designs with unequal

replications. It was shown that if N is a BIBD in treatments 1, 2, ...,v with parameters v, b, r, k, λ by adding to this design v blocks such that the i^{th} contains a new treatment zero and the treatment i, (k-1) times, the new design N_1 in (v+1) treatments will be balanced k-ary design provided r=(v-1) and λ=(k-1).

There are situations in which the available animals cannot be used completely for the experiment using conventional designs. In such circumstances also it is easy to construct balanced designs with equal replications and unequal block sizes.

If there is a randomised block design with v treatments 1, 2,....,v, then, it was shown that, the design N_1 obtained by adding v blocks each of size 2, the i^{th} containing a new treatment zero and the old treatment i, will be balanced binary design for (v+1) treatments with different block sizes and different replications provided $r= \frac{v}{2}$.

Introduction of Kronecker product for the construction of proper equireplicate balanced design is one of the outstanding features of the thesis. The following results were established.

Let N_1 and N_2 be two BIB designs with parameters v, b_1, r_1, k_1, λ_1 and v, b_2, r_2, k_2, λ_2 respectively. For positive integral values of

a_1 and a_2, $a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)$ is in general a proper equireplicate n-ary design provided $a_1+a_2+1=n$.

Taking $a_1=a_2=1$ that $E(1,b_2)xN_1+N_2xE(1,b_1)=N$ is a ternary proper equireplicate design with number of blocks b_1b_2, number of treatments v, $r= b_1r_2+b_2r_1$ and $k=k_1+k_2$.

If $a_1=1$, $a_2=n-2$ and $N_2=I(v)$ a n-ary design with $b=b_1v$ $r=r_1v+b_1$ $k=k_1+1$ and number of treatments equal to v is obtained.

N_1 and N_2 are two balanced proper equireplicate n_1-ary and n_2-ary designs in v treatments with b_1, b_2 blocks respectively. If a_1 and a_2 are positive integers $a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)$ is a n-ary balanced equireplicate proper design with b_1b_2 blocks where $n=a_1(n_1-1)+a_2(n_2-1)+1$.

If $a_1=a_2=1$, $N_1=N_2$ a t-ary balanced equireplicate proper design then N is a proper equireplicate balanced (2t-1)-ary design.

If $a_1=a_2=1$, N_1 a p-ary design and N_2 a s-ary design then N is a p+s-1-ary design.

The importance of n-ary design is that the intrablock analysis is pretty easy.

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BALANCED N-ARY DESIGNS WITH EQUAL OR UNEQUAL BLOCK SIZES AND EQUAL OR UNEQUAL REPLICATIONS

BY

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ABSTRACT OF A THESIS

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ABSTRACT

Tocher (1952) introduced n-ary designs as generalisation of balanced incomplete block designs. But the properties of the parameters of the design have not been discussed so far. We have shown that some important properties of the balanced incomplete block binary design are also true in the case of balanced n-ary symmetrical proper equireplicate designs.

That is if $h = \sum_j n_{ij}^2$; $\lambda = \sum_j n_{ij} n_{pj}$; in a proper equireplicate balanced design then

(i) $h > \lambda$

(ii) $b \geq v$

(iii) $rk = h + (v-1)\lambda$

Among the methods block section, block intersection, complementation and inversion considered by us for the construction of designs the method of complementation is only found fruitful for the construction of proper equireplicate balanced designs.

There are situations like comparison of new varieties of seeds of which are in short supply where equal replication of treatments is not possible. There may also be contexts in which the available few animals cannot be used completely for the experiment using conventional designs. For such circumstances we have

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proposed a systematic method of construction of balanced n-ary designs with equal or unequal replications and equal or unequal block sizes.

The method of Kronecker product has been formally introduced to the literature for the construction of proper equireplicate balanced n-ary designs, and the methods is contained in the following results.

If N_1 and N_2 are two BIB designs with parameters v, b_1, r_1, k_1, λ_1 and v, b_2, r_2, k_2, λ_2 respectively, for positive integral values of
a_1 and a_2, $a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)$ is in general a proper equireplicate n-ary design provided $a_1+a_2+1 = n$.

If N_1 and N_2 are two balanced proper equireplicate n_1-ary and n_2-ary designs in v treatments with b_1, b_2 blocks respectively, for positive integers
a_1 and a_2, $a_1E(1,b_2)xN_1+a_2N_2xE(1,b_1)$ is a n-ary balanced equireplicate proper design with b_1b_2 blocks where
$n=a_1(n_1-1)+a_2(n_2-1)+1$.

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